IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

TOPOLOGY OF THE UNIVERSE

[1]Sowmiya.R ,[2]Senbaga Priya.K,[3]Selva Nandhini.N

[1,2,3]Post Graduate,Department of Mathematics,Sri Krishna Arts and Science College, Coimbatore,TamilNadu,India

Abstract –

The great charm resulting from this consideration (a multi-connected universe) lies in recognition of the fact that the universe of these being is finite and yet has no limits.

- *Albert Einstein*

One of the finest question arise for everyone's space curiosity is What is the actual shape of the universe?, If the universe has boundary or not ?, Is it finite or infinite?. And the global topology of the universe which including the question also about its volume and its connectedness. It is the fundamental issue in cosmology which has been overlooked for many years and it also proved by many scientist. In this paper, I just reviewed briefly the historical background of the Cosmic Topology. And I extended the shape of the universe that is predicted in the Fourth dimension.

Birth of the Cosmological Topology:

One of the oldest cosmological question is the physical extension of space: is it finite or finite? (see e.g. LUMINET & LACHIEZ- REY ,1994). In the history of cosmology, it is well known that the Newtonian physical space, mathematically identified with infinite Euclidean space $\mathbb{R}^3$, gave rise to paradoxes such as darkness of night (see HARISON, 1987) and to problems of boundary conditions. Regarding for instance the Mach's idea according to which local inertia would result from the contributions of masses at infinity, an obvious divergence difficulty arose, since a homogeneous Newtonian universe with non-zero density had an infinite mass.

The aim of relativistic cosmology was to deduce from gravitational field equations some physical models of the universe as a whole. When Einstein (1917) assumed in his static cosmological solution that space was to provide a model for a finite space, although without a boundary. He regarded the closure of space as necessary to solve the problem of inertia (EINSTEIN 1934). The spherical model cleared up most of the paradoxes stemmed from Newtonian cosmology in such an elegant way that most cosmologist of the time adopted the new paradigm of a closed space, without examining other geometrical possibilities. Einstein was also convinced that the hypersphere provided not only the metric of cosmic space namely its topology. However, topology does not seem to been a major preoccupation of EINSTEIN: his 1917 cosmological article did not mention any topological alternative to the spherical space model.

Some of his colleagues pointed out to Einstein the arbitrariness of his choice. The reason was the following, The global shape of shape is not only depending on the metric; it primarily depends on its topology and requires a complementary approach to Riemannian differential geometry. Since Einstein's equations are partial derivative equations, they describe only local geometrical properties of space-time. The latter are contained in a metric tensor, which enables us to calculate the components of the curvature tensor at any non singular point of space-time: to give metric solution of the field equations, correspond several (and in most cases an infinite number of) topologically distinct universe models .

First , DE SITTER (1917) noticed that the Einsten's solution admitted a different spaceform, namely the 3-dimensional projective space (also called elliptic space), obtained from the hypersphere by identifying antipodal points. The projective space has the same metric as the spherical one, but a different topology (for instance, for the same curvature radius its volume is twice smaller).

Next H. Weyl pointed out the freedom of choice between spherical and elliptical topologies. The EINSTEIN'S answer (1918) was unequivocal: "Nevertheless I have like an obscure feeling which leads me to prefer the spherical model. I have the presentiment that manifolds in which any closed curve can be continuously contracted to a point are the simplest ones. Other persons must share feeling, otherwise astronomers would have taken into account the case where our space is Euclidean and finite. Then the two- dimensional Euclidean space would have the connectivity properties of a ring surface. It is an Euclidean plane in which any phenomenon is doubly periodic, where points; located in the same periodical grid are identical. In finite Euclidean space, three classes of non continuously contractible loops would exist. In a similar way, the elliptical space possesses a class of non continuously contractible loops, contrary to the spherical case, it is the reason why I like it less than the spherical space. Can it be proved that elliptical space is the only variant of spherical space? It seems yes to me".

EINSTEIN (1919) repeated his argumentation in a postcard sent to Felix Klein: "I would like to give a reason why the spherical case should be preferred to the elliptical case. In spherical space, any closed curve can be continuously contracted to a point, but not in the elliptical space; in other words the spherical space alone is simply-connected, not the elliptical one […] Finite spaces of arbitrary volume with the Euclidean metric element undoubtedly exist which can be obtained from the infinite spaces by assuming a triple periodicity, namely identity between sets of points. However such possibilities, which are not taken into account by general relativity, have the wrong property to be multiply-connected". From these remarks it follows that the Einstein's prejudice in favour of simple-connectedness of space was of an aesthetical nature, rather than being based on physical reasoning.

In his answer to Weyl, Einstein was definitely wrong on the last point: in dimension three, an infinite number of topological variants of the spherical space- all closed –do exist, including the so-called lens spaces(whereas in dimension two, only two spherical spaceforms exist, the ordinary sphere and the elliptic plane). However, nobody knew this result in the 1920's: the topological classification of 3-dimensional spaces was still at its beginnings. The study of Euclidean spaceforms started in the context of crystallography. FEOFOROFF (1885) classified the 18 symmetry groups of crytstalline structures in R^3 *, BIEBERBACH (1911) developed a full theory of crystallographic groups, and twenty years later only NOWACKI (1934) showed how the Bieber bach's results could be applied to achieve the classification of 3-dimensisonal Euclidean spaceforms. The case of spherical spaceforms was first set by KLEIN (1890) and KILLING (d1891). The problem was fully solved much later (Wolf, 1960). Eventually, the classification of homogeneous hyperbolic research (THURSTON, 1979, 1997).*

Going back to relativistic cosmology, the discovery of non-static solution by FRIEDMANN(1922) and, independently, LEMAITRE (1927), opened a new era for models of the universe as a whole (see, e.g., LUMINET, 1997 for an epistemological analysis). Although Friedmann and Lemaitre are generally considered as the discoverers of the big bang concept- at least of the notion of a dynamically universe evolving from an initial singularity – one of their most original considerations, devoted of the topology of the space, was overlooked. As they stated, the homogeneous isotropic universe models (F-L models) admit spherical, Euclidean or hyperbolic space sections according to the sign of their (constant) curvature (respectively positive, zero or negative). In adding, FRIEDMANN (1923) pointed out the topological indeterminacy of the solution in his popular book on general relativity, and he emphasized how the Einstein's theory unable to deal with the global structure of the space-time. He gave simple example of the cylinder – a locally Euclidean surface which has not the same topology as the plane. Generalizing the argument to higher dimensions, he concluded that several topological spaces could be used to describe the same solution of Einstein's equations.

Topological considerations were fully developed in his second cosmological article (FRIEDMANN, 1924), although primarily devoted to the analysis of hyperbolic solutions. FRIEDMANN clearly outlined the fundamental limitations of relativistic cosmology: "without additional assumptions, the Einstein's world equations do not answer the question of the finiteness of our universe", he wrote. Then he described how the space could become finite (and multi-connected) by suitably identifying points. He also predicted the possible existence of "ghost" images of astronomical sources, since at the same point of a multi-connected space an object and its ghost would coexist. He added that "a space with positive curvature is always finite ", but he recognized that fact that the mathematical knowledge of his time did not allow him to "solve the question of finiteness for a negatively- curved space".

Comparing with Einstein's reasoning, it appears the cosmologist had no prejudice in favour of a simply-connected topology. Certainly Friedmann believed that only spaces with finite volume were physically realistic, Prior to his discovery of hyperbolic solutions, the cosmological solutions derived by Einstein, de sitter and himself had a positive spatial curvature , thus a finite volume. With negatively – curved spaces, the situation became problematic, because the "natural" topology of hyperbolic space has an infinite volume. It is the reason why friedmann , in order to justify the physical patience pertinence of his solutions, emphasized the possibility of compactifying space by suitable identifications of points.

Lemaitre fully shared the common belief in the closure of space. In a talk given at the institute Catholique de Paris (LEMAITRE, 1978), the Belgian physicst expressed his view that Riemannian geometry "dissipated the nightmare of infinite space". His two major cosmological models (the non-singular, "Eddington-Lemaitre" model, 1927, and the singular, "hesitating universe" model, 1931) assumed positive space curvature. Thus Lemaitre thoroughly discussed the possibility of elliptical space, that he preferred to the spherical one. Later, LEMAITTRE (1958) also noticed the possibility of hyperbolic spaces as well as Euclidean spaces with finite volumes for describing the physical universe.

Such fruitful ideas of cosmic topology remained widely ignored by the main stream of big bang cosmology. Perhaps the EINSTIEN-DE SITTER model (1932), which assumed Euclidean space and eluded the topological question, had a negative influence on the development of the field. Almost all subsequent textbooks and monographies on the relativistic cosmology assumed that the global structure of the infinite Euclidean space, or the infinite hyperbolic space, without mentioning at all the topological indeterminacy. As a consequence, some confusion settled down about the real meaning of the terms "open" and "closed" the F-L solutions, even in recent specialized articles (eg. WHITE and SCOTT . 1906) whereas they apply correctly to time evolution (open models stand for every expanding universe, closed models stand for expanding- contracting solutions), they do not properly describe the extension of the space (open for infinite , closed for finite). Nevertheless it is still frequent to read that the (closed) spherical model have infinite volumes. The correspondence is true only in the very special case of a simply – connected topology and a zero cosmological constant. According to Friedmann's original remark, in order to know if a space is finite or infinite, it is not sufficient to determine the sign of its spatial curvature, r equivalently in a cosmological context to measure the ratio of the average density to the critical value: additional assumptions are necessary – those arising from topology, precisely.

Until 1995, investigations in cosmic topology were rather scarce (see references in LaLu95). From an epistemological point of view, it seems that the prejudice in favour of simply- connected (rather than multi-connected) spaces was of the same kind as the prejudices in favour of static (rather than dynamically) cosmologies during the 1920's. At the first glance, an economy principle" (often useful in physical modeling) could be invoked to preferably select the simply-connected topologies. However, on one hand, new approaches of spacetime, such as quantum cosmology, suggest that the smallest closed hyperbolic manifolds are favored (CORNISH, GIBBNS & WEEKS, 1908), thus providing a new paradigm for what is the "simplest" manifold. On the other hand, present astronomical data indicate that the average density of the observable universe is less than the critical value ($\Omega = 0.3 - 0.4$), thus suggesting that we live in a negatively-curved F-L universe. Putting together these two requisites, cosmologists must face that fact that a negatively- curved space with a finite volume is necessarily multi-connected.

FURTHER DEVLOPMENTS:

In the last decades, much effort in observational and theoretical cosmology has been directed towards determining towards determining the curvature of the universe. The problem of topology of space-time was generally ignored within the framework of classical relativistic cosmology. It began to be seriously discussed in quantum gravity for various reasons: the spontaneous birth of the universe from quantum vacuum requires the universe to have compact spacelike hypersurfaces, and the closure of space is a necessary condition. To render tractable the integrals of quantum gravity (ATKATZ & PAGELS,1982). However, the topology of space-time also enters in a fundamental way in classical general relativity. Many cosmologists were surprisingly unaware of how topology and cosmology could fit together to provide new insights in universe models. Aimed to create a new interest in the field of cosmic topology, the existensive review by LaLu95 stressed on what multi-connectedness of the universe would mean and on its observational consequences. However two different papers (STEVENS et al.,1993; DE LIVEIRA COSTA & SMOOT,1995) declared that the small universe idea was "dead"; in fact, drawing general conclusions from few examples mostly taken into Euclidean case, they did not take into account the most interesting spaces for realistic universe models, namely the compact hyperbolic manifolds, which require a quite different treatment (LaLu95, CORNISH et al., 1997a). Ironically enough, a worldwide interest for the subject has flourished since 1995, both from an observational point of view and from a theoretical one: approximately the same amount of papers in cosmic topology have been published within the last 3 years as in the previous 80 years! Interesting progress has been achieved in mathematics as well as in cosmology. I briefly summarizes below some of these advances.

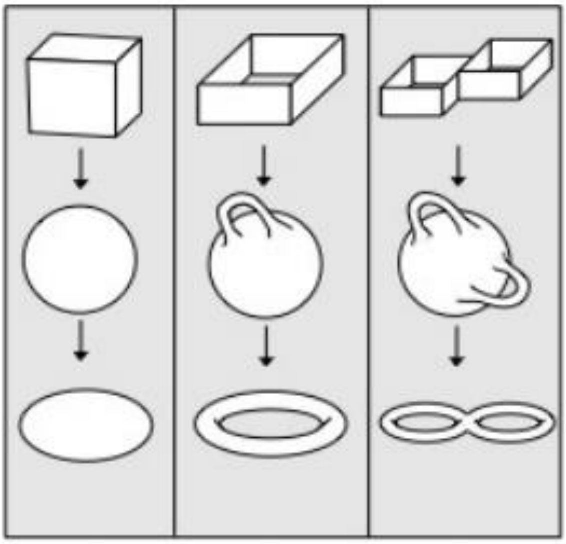
MATHEMATICAL ADVANCES :

Topological classification of the spaces:

Topology can be defined as the study of continuous transformations, albeit the property which remains unchanged when continuous transformations are made on a geometry, like squeezing, stretching etc. which change its metric but not its topology. Two manifolds belong to the same topological class are called homeomorphic and can be continuously and reversibly transformed into each other. In other words, if we have two manifolds M_1 *and* M_2*, a homemorphism between them would be continuous map* $\varphi: M_1 \leftrightarrow M_2$*with a defined inverse.*

FLRW Models admit spatial sections of homogeneous and isotropic spherical, hyperbolic or Euclidean manifolds depending on whether the sign of spatial curvature is positive, negative or zero. However, there is often a common misconception that flat or hyperbolic universes imply an infinite universe, which was proved unfounded long ago by Friedmann and Lemaitre who discovered that FL metrics with zero or hyperbolic topologies did admit spatially closed topologies. However, the works of these and many other people remain ignored and cosmology textbooks implicitly assume space to be a simply connected hypersphere.

Given the isotropy of the microwave background, it is implied that the curvature of space is almost constant throughout. Hence literature on possible manifolds for the universe focuses mainly on manifolds of constant curvature. General relativity is invariant under diffeomorphisms which signify change of coordinates but not homeomorphisms. Thus, the principle of covariance goes a long way in predicting the laws of physics that a body follows in different regions of space , however topology remains independent of such a correlation.



Below are explained some basic concepts that are frequently used in classifying and studying topological spaces

- ***Homogeneous Spaces:***

For two dimensional surfaces, it was shown by, that if a space is closed and connected, it is homeomorphic to Riemannian surfaces of constant curvature. A Riemann surface is defined as a complex manifold of dimension one. Hence all closed surfaces can be classified into one of the three Riemannian metrics: Spherical S^2*, Euclidean* $\boldsymbol{E^2}$ *and Hyperbolic* $\boldsymbol{H^2}$*.*

In three dimensions, this is not true, as we can clearly see from a three dimensional cylinder$\boldsymbol{S^2} \times \boldsymbol{R}$ *which is not homeomorphic to any of the constant curvature geometries, neither the spherical nor the Euclidean manifold. The metric of the three dimensional cylinder is homogeneous but an isotropic. There are in total eight types of homogeneous three dimensional geometries out of which only five of them are of a constant curvature.*

The symmetries of a manifold can be quantified with the group G of isometries, which are transformations to the manifold that leave the metric invariant. A homogeneous manifold, G is non-trivial. The group H of all points y acts transitively on M as $g \in G such\ that g(x) = y$ *with y being referred to as the orbit of x. The subgroup of isometries that leave the point x fixed (for instance, rotations around x) is called the isotropy group, I at x. These isometries are related by*

$$dim(G) = dim(H) + dim(I).$$

G is simply transitive on H if $dim(G) = dim(H)$ *and multiply transitive if dim(G)> dim(H).*

The isometry group has dimensions $\leq n(n+1)/2$ for an n-dimensional manifold and attains maximum value for a maximally symmetric manifold. A maximally symmetric manifold is essentially a manifold which has the same number of symmetries as an ordinary Euclidean space. For the space-time metric of our universe, a maximally symmetric space should have an Isometry group of dimension = 10.

Analyses, space times of dimensions $(\geq 6 \leq 10)$ are not realistic cos-mological models due the large number of dimensions involved. For dim (G)≤6,the group may act on M or lower dimensional manifolds. For dim(G) = 4, the isometry acts on a manifold that is homogeneous in space and time and gives us a model which is spatially symmetric in space and time, but does not allow expansion. For the alternative case, when a subgroup of G acts only on the space-like hypersurfaces, giving us spatially homogeneous space times. This scenario has three possible sub cases:

1) dim(G) = 6, where G3 acts on spatially homogeneous spaces and there is a G^3isotropy group. Thus, we have spatially homogeneous and isotropic spacetimes which also allow space like hypersurfaces of constant curvature i.e. the FLRW models.

2) dim(G) = 3, where there is only one group of isometries , i.e. the group G3 acting on the spatially homogeneous spaces. Thus we have homogeneous and an isotropic spaces, one example of which are the Bianchi models [20].

3) dim (G) = 4, where G is multiply transitive on 3 dimensional subspaces, some such space times are discussed in [21]. We will not consider these space-times in this text.

- ***Simply and Multiply Connected Spaces:***

A good place to begin is the definition of the concept of homotopy which is an important classification regime used in topology. Two loops γ and γ' drawn on a manifold surface are said to be homotopic if one can be continuously trans-formed to the other. A simply connected manifold now can simply be defined as a manifold for which any loop is homotopic to another, or equivalently, all loops are homotopic to a point. If this condition is not true for all possible loops on the manifold, it is multi-connected. Homotopic loops give us information about holes or handles in a manifold. In higher dimensions, one dimensional homotopy loops are not enough to encompass all the properties of the topology, leading to the introduction of homotopy groups. The first homotopy group is called the fundamental group. Poincare conjectured that any connected closed n-dimensional manifold with a trivial fundamental group is topologically equivalent to a sphere.

Multi connectedness implies that the fundamental group is non-trivial, essentially meaning that there is one hole in the manifold. Poincar'e in 1904, conjectured that a connected closed n-dimensional manifold with a trivial fundamental group must be topologically equivalent to a sphere, S^n .

The set of solutions to Einstein's equations does not place any topological constraints on the manifold except its curvature. The FLRW models describe the observed universe with the greatest accuracy among the known models and give solutions for homogeneous and isotropic models with spherical, hyperbolic or flat topologies further incorporating a wide variety of possible solutions like the desitter solution or solutions with a cosmological constant or a non standard equation of state. The assumption in most literature of a simply connected universe is arbitrary and replacing the same with a multi-connected universe changes avery few characteristics in the FLRW models. One of the differences lies in the range of the coordinates where for a simply connected universe, one would have $\varphi: 0 \to 2\pi, \theta: -\pi/2 \to \pi/2,\ \chi: 0 \to \infty\ for\ k = -1, 0\ and\ 0 \to \pi\ for\ k = 1$ whereas for a multi-connected model, space is smaller and the range of the coordinates is reduced. As discussed earlier, observations suggest that the universe is homogeneous and locally isotropic which implies that space has constant curvature. Thereby, most multi-connected models explored in literature rely on this assumption with the exception of Bianchi and Lemaitre-Bondi models among a few others. How-ever, even with the anisotropic models, the homogeneity and local isotropy of these models ensure that the CMB remains isotropic. A significant difference19 however is observed in the spectrum of density fluctuations.

While the finiteness of simply connected models can simply be determined from the sign of curvature of the manifold, i.e. infinite for $k = 0, -1$ and finite for $k = 1$, the same does not hold true for multi-connected topologies. As early as 1924, it was known that multi-connected models with a zero or negative curvature admitted spatially closed topologies . For instance a toroidal universe is of a finite volume and circumference despite zero curvature.

- ***Fundamental Domain:***

A simple example is a torus whose fundamental domain is a rectangle. To obtain a torus from a rectangle, we first identify one pair of opposite sides in the rectangle, thereby getting a cylindrical tube. Identifying the other pair of opposite sides gives us a torus. The transformations done in identifying the opposite edges form a holonomy group. A holonomy group is a subset of the full isometry group of the covering space. To understand the holonomy group, consider a point x and a loop γ at x in the manifold M. If γ lies in a simply connected domain of M, it generates a single point $\tilde{x}$ in $\widetilde{M}$ but if the manifold is multi-connected, it creates a set of points $\widetilde{x'}, \widetilde{x''}$ ... which are said to be homologous to $\tilde{x}$. The displacements form isometries referred to as the holonomy group Γ in $\widetilde{M}$. Since there is a non zero distance between the homologous points, the group is *discontinuous and has no fixed generating point. Thus, the holonomy group is said to act freely and discontinuously on* $\tilde{M}$.

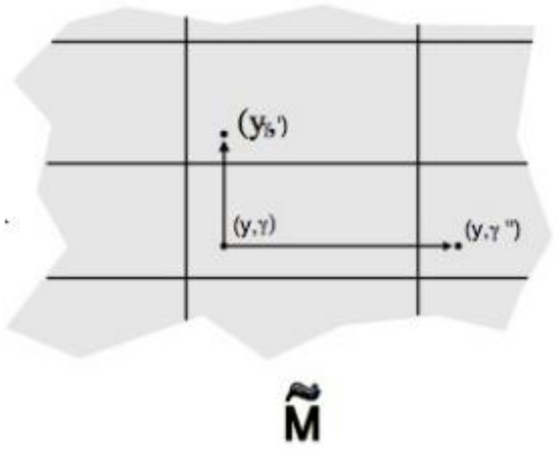
The full isometries of the universal cover are broken by identifications and we can represent a compact manifold as a quotient space given by G/Γ *where G is the group of isometries of the domain and* Γ *is the holonomy group.*

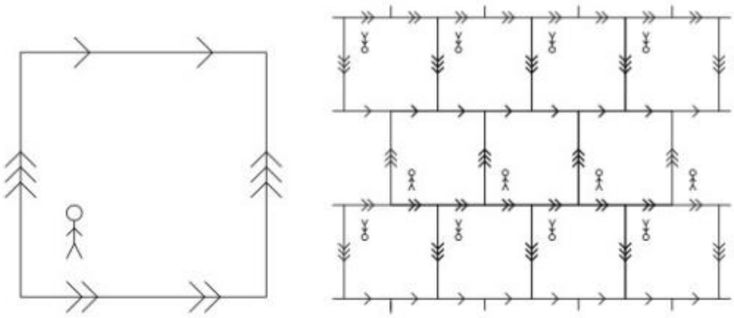
- ***Universal Covering Space:***

The universal cover of a connected topological space is a simply connected space with a map $f: Y \rightarrow X$ *that is a covering map. By acting with the transformation group on the fundamental domain, we get many identical copies of the domain which give us the universal covering manifold. In case of simply connected spaces this universal covering is identical to the fundamental domain for instance a sphere* S^2 *is its own universal cover, however in the case of multiply connected spaces we get replicas of the central manifold. In this case, a universal manifold is constructed as follows. The space is cut open to make a simply connected space with edges, called the fundamental domain of the manifold. For instance, a hyperbolic octagon for a double torus and a square for a square torus. Now add another copy of the fundamental domain to the edge and keep doing so until all edges of the original manifold are covered. More copies are added to the resulting space recursively until a covering map with possibly infinite number of copies of the fundamental domain is obtained. The largest such possible cover is called the universal covering space. Thus, If* $f : Y \rightarrow X$ *is a covering map, then there exists a covering map* $f: \tilde{X} \rightarrow Y$ *such that the composition of* X *and* $\tilde{X}$ *is the projection from the universal cover to* X.

For example, a flat torus tiling a universal covering space can be likened to the screen of video game where on walking off the right end of the screen, one would emerge on the opposite edge and the same for the vertical edges. Thus, one gets the impression of an infinite space even though it is just a repetition of the same fundamental domain over and over. This type of a universal covering space is constructed by identifying the edges of a fundamental domain, and identifying the edges differently gives rise to a different set of orientations and symmetries on transition between different universes.

1)





- ***Detectability of a Multi Connected Topology:***

There are basically three possible correlations between detectability of topology and the size of the universe. First, that the universe could be infinite in which case, it is not possible to detect topology with currently known methods. Secondly the universe is finite but much larger than the scale of the observable universe in which case, it is hard to detect visible signs of topology. The last and the best scenario would be a universe which is finite and comparable to the size of the observable universe, where we can thus use current methods to detect its structure.

For a manifold M, there can be defined the smallest and largest circles in scribable in M, r_{inj}, described in terms of the smallest closed geodesic, $l_m$andr_{max} respectively. A closed geodesic, that passes through a point x, in a multiplyconnected manifold is a part of the geodesic that connect that point to its image in the covering space $\widetilde{M}$. The length of any such closed geodesic which passes through x, in a manifold with a fixed isometry g, is given by the distance function:

$$\delta g(x) = d(x, gx)$$

In terms of this distance, the injectivity radius can be defined as

$$r_{inj}(x) = \frac{1}{2} min_{g\in \Gamma}\{\delta g(x)\}$$

where $\tilde{\Gamma}$ denotes the covering group without the identity map. We can define the observation survey depths to be χ_{obs} . A topology is said to be undetectable if $\chi_{obs} < r_{inj}$ in which case we cannot detect any multiple images in the observable sky, and detectable for $\chi_{obs} > r_{inj}$

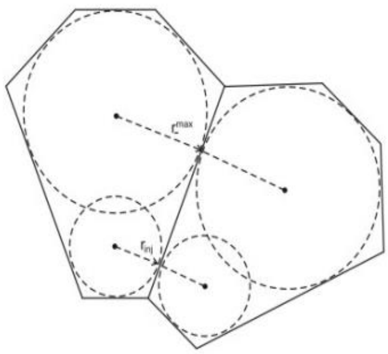
In a globally homogeneous manifold, the distance function for any covering isometry is constant, hence r_{inj} is constant throughout space and the detectability of a topology does not depend on the observers position in the manifold, In inhomogeneous manifolds however, r_{inj} varies from point to point and thus the topology depends on both the observers position and the survey depth. In this case however, we can still define an absolute undetectability condition, that is if for

$$r_{inj} = min_{x\epsilon M}\{r_{inj}(x)\},$$

$$\chi_{obs} < r_{inj}$$

then the topology is undetectable for all observers in M

For a flat manifoldE3, the relationship between the horizon radius and the injectivity radius is pretty arbitrary since it is possible to stretch its translational components to obtain any injectivity radius. It is thus probably least likely to be able to detect a Euclidean topology from the three possible topologies. On the other hand, in the case of a Hyperbolic topology, the volume of the fundamental domain increases as the complexity of the Hyperbolic group Γ increases. However,the injectivity radius for the smallest hyperbolic manifolds exceeds the horizon radius. Spherical manifolds on the other hand decrease in size as the symmetry group Γ becomes larger.



Three Dimensional Manifolds of Constant Curvature:

Current observations of the observable part of the universe imply a homogeneous and isotropic geometry to a precision of 1 part in 10^4, thus we should begin by considering topologies that are locally homogenous and isotropic, thereby the constant curvature Euclidean, Spherical and Hyperbolic geometries as deduced in this paper.

Any compact 3-manifoldMwith a constant curvature k allows a discrete isometry subgroup Γ acting freely and discontinuously on $\breve{M}$. Such a manifold can also be written as$\breve{M}/\Gamma$, where $\breve{M}$ is the universal covering space of M, given by Euclidean space ($for\ k = 0$), 3-sphere (for $k > 0$) or the hyperbolic 3-space($for\ k < 0$).

does an excellent work of classifying the Euclidean, Spherical and Hyperbolic manifolds further into sub manifolds as discussed below.

- ***Euclidean Manifold:***

The line element of the Euclidean covering space is given by:

$$d\sigma^2 = R^2\{d_\chi{}^2 + \chi^2(d\theta^2 + sin^2\theta d_\phi{}^2)\}$$

Where $\chi \geq 0$.

The full isometry group is given by $G = ISO(3) = R^3 \times SO(3)$ *and the generators of the possible holonomy groups* Γ *include different combinations of identity, translations, glide reflections and helicoidal motions. In total 18 different types Euclidean manifolds can be generated. The manifolds can be classified primarily into open and closed models.*

The open models include orientable and non-orientable space-forms which can be classified with glide reflections. On excluding glide reflection as a holonomy group, we get four orientable space-forms. The non-orientable space forms are not relevant to cosmology. The closed models on the other hand can be classified according to the different possible ways the opposite faces of a parallelepiped can be identified with each other. Another class of identifications can also be made on hexagonal fundamental polyhedron with rotations of 2π/3 and π/3.

- ***Spherical Manifolds:***

Spherical manifolds have a universal covering of a compact hypersphere. A 3-sphere S^3 *of radius R is the set of all points in 4-Dimensional Euclidean Space.*
The metric of the 3-sphere with coordinates $x^0, x^1 x^2, x^3$ *can be written as*

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = R^2$$

Converting to angular coordinates(χ, θ, φ), *for* χ *and* θ: $[0, \pi]$, φ: $[0,2\pi]$ $x^0 = R\ cos\chi$, $x^1 = R\ sin\chi\ cos\theta$, $x^2 = R\ sin\chi sin\theta cos\varphi$, $x^2 = Rsin\chi\ sin\theta\ sin\varphi$.
We get the metric:

$$d\sigma^2 = R^2\{d_\chi{}^2 + sin_\chi{}^2(d\theta^2 + sin^2\theta d_\phi{}^2)\}$$

From

$$d\sigma^2 = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

The volume of the covering manifold is given by

$$vol(S^3) = \int_0^\pi 4\pi\ R^2\ sin_\chi^2\ Rd_\chi = 2\pi^2 R^3$$

Substituting $r = sin\chi$ *in metric, we get the FLRW metric form of the spherical manifold*

$$d\sigma^2 = R^2\{\frac{dr^2}{(1-r^2)} + r^2(d\theta^2 + sin^2\theta d_\phi{}^2)\}$$

A good way of visualizing a 3-sphere is to use the analogy of a 2-sphere,where if we intersect the sphere with a plane and pass it through the sphere, the intersection points form circles of increasing diameter and subsequently of de-creasing diameter. Similarly, one can imagine the intersection of a 3-sphere with a 3-dimensional hyperspace and forming spheres of increasing diameter before reducing in size again to zero.

The holonomy groups of S^3 *were classified into subgroups* Γ *of* $SO(4)$ *acting freely and discontinuously on* S^3*:-*

1) *Cyclic group of order* p, Zp $(p \geq 2)$: Zp *can be seen as generated by the rotations by an angle* $2\pi/p$ *about some axis* $[\theta, \varphi]$ *of* R^3.

2) Dihedral group of order $2m, Dm(m \geq 2)$: Generated by rotations in the plane by an angle $2\pi/m$ and a reflection about a line through the origin. The operation preserves regular m-gons lying in the plane and centered on the origin.

3) Polyhedral Groups: Symmetry groups of the regular polyhedra in R^3 namely the Tetrahedral group of order 12, octahedral group of order 24 and Icosahedral group of order 60. The cube is included in the symmetry group of the octahedron and the dodecahedron is included in the symmetry group of the icosahedron.

There is an infinite number of spaces that can be obtained by taking the quotent of S^3 with the above groups and varying the parameters p and m. The volume of the quotient manifold, $M = S3/\Gamma$ obtained is given by

$Vol\ (M) = 2\pi^2 R^3/|\Gamma|$.

- ***Hyperbolic Manifolds:***

Some of the most important contributions to locally hyperbolic spaces were made by Thurston , however these manifolds still remain much less understood than other homogeneous manifolds. Nevertheless,H^3 can be embedded in Minkowski space, R^3 of metric

$$ds^2 = -(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$$

as the hypersurface

$$-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$$

Thereby, the generators of the fundamental group G of H^3can be related to homogeneous Lorentz transformations.

Let us make coordinate transformations to introduce (χ, θ, φ) with $\chi \in [0, inf), \theta \in [0, \pi], \varphi \in [0,2\pi]$, we get

$$x1 = Rcosh\chi, x2 = Rsinh\chi cos\theta, x3 = Rsinh\chi sin\theta cos\varphi, x4 = Rsinh\chi sin\theta sin\varphi.$$

We thereby get the metric for H^3 as:

$$d\sigma^2 = R^3\{d_\chi{}^2 + sinh^2{}_\chi(d\theta^2 + sin^2\theta d\phi^2)$$

This metric can be expressed in a more commonly used, FLRW form, of the metric, obtained by the coordinate change, $r = sinh\chi$ that gives us:

$$d\sigma^2 = R^2\{\frac{dr^2}{(1 + r^2)} + dr^2\left(d\theta^2 + sin^2\theta d_\phi{}^2\right)\}$$

The holonomies of H^3can be described as the group of fractional linear trans-formations acting of the complex plane:

$$z' = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{C}, ad - bc = 1.$$

This group is equivalent to the group of conformal transformations of R^3 which leave the upper half space invariant.

The hyperbolic geometries for dimension > 2 are different from the 2-dimensional case. For instance, while a surface $(genus \geq 2)$ can support an infinite number of non-equivalent hyperbolic metrics, a connected oriented manifold $(n \geq 3)$ can only support at most one hyperbolic metric. This is confirmed further by the rigidity theorem which confirms that if two hyperbolic manifolds (dimension, $n \geq 3$) have isomorphic fundamental groups, they are necessarily isometric to each other. Hence for $n \geq 3$, the volume of a manifold and the lengths of its closed geodesics are topological invariant

For compact Euclidean spaces, the fundamental polyhedron can possess only a maximum number of eight faces, despite allowance of arbitrary volume. In spherical manifolds, the volume needs to be finite and a fraction of the maximum possible volume S, i.e. S/Γ. For the case of hyperbolic manifolds however, there is no limit on the possible number of faces of the fundamental polyhedron. There is a lower limit however, on the minimum volume of the hyperbolic 3-manifold,a lower limit of which was set by Meyerhoff to be

$Vol_{min} > 0.00082R3$.

COSMOLOGY TOPOLOGY

The global topology of the universe can be tested by studying the 3-Dimension distribution of discrete sources and the 2-Dimension fluctuations in the "COSMIC MICROWAVE BACKGROUND" (CMB). This methods, which is based on the "ghost image" which was predicted by FRIEDMANN(1924), namely topological images and it is same as celestial object such as galaxy, a cluster or a spot in the CMB (the term "ghost" can lead to confusion in the sense that all the images are on the same foot of the reality). Such topological images can appear in a multi-connected space a characteristic length scale of which is smaller than the size of the observable space, because light emitted by a distant source can reach an observer along several null geodesics. In the 1970's systematic 2-D observations of galaxies, undertaken at the 6-m Zelentchuk telescope under the supervision Schvartsman, allowed to fix lower limits to the size of physical space as 500^{-1}*Mpc (see LaLu95 and references herein). A new observational test based on the 3-D analysis of cluster seperations, the so-called "cosmic crystallographic method", has been proposed by LEHUCQ, LACHIEZE REY & LUMINET (1996), and further discussed in the literature (FAGUANDES & GAUSMANN, 1997; ROUKEMA & BLANLOEIL,1998). Other 3-D methods, using special quasars configurations (ROUKEMA,1996) or X-ray clusters (ROUKEMA & EDGE,1997), have also devised , see ROUKEMA (1998) for a summary. However, the poorness of 3-D data presently limits the power of such methods.*

Some authors (DE OLIVEIRA-COSTA et al., 1996), still believing that an inflationary scenario necessarily leads to an Einstein-de-Sitter universe, looked for constraints on topology with the CMB by investigating the compact Euclidean manifolds (CEM) only. They found that toroidal universe with rectangular cells (the simplest CEM, describe as E_1 *in LaLu95's (classification), with cell size smaller than* $300h^{-1}$ *Mpc for a scale-invariant power spectrum, were ruled out as "interesting cosmological models". However, as shown by FAGUNDES & GAUSMANN (1997),CEMs remain physically meaningful even if the size of their spatial sections is of the same order of magnitude as the radius of the observable horizon. Using the method of cosmic cryptallography (LEHOUCQ et al ., 1996) they performed simulations showing sharp peaks in the distributions of distances between topological images.*

In any case, CHMs appear today as the most promising specimens for cosmology, both from theoretical and observational grounds. Topological signature using the "circles in the sky" method (CORNISH et al., 1996b) is difficult to detect in COBE data, but it could be possible with the future MAP and /or PLANCK data.

In fact, full cosmological calculations in CHMs (e.g. simulations of CMB fluctuations, or possible Casimir-like effects in the early universe) are difficult; they require calculations of Eigenmodes of the Laplace operator acting on the compact manifold. The problem is not solved. CORNISH & TUROK (1998) recently worked out the method in 2-D, but the 3-D case could reveal less tractable. Compactness renders the calculations more difficult due to "geodesics mixing", namely chaotic behavior of geodesic bundle (CORNISH et al., 1996a). Some authors (LEVIN et al ., 1997, CORNISH et al., 1997) were able to perform calculations in the "horn topology", an open hyperbolic space introduced by SOKOLOFF and SAROBINSKY (1976), but the essential job remains to be done.

Another underdeveloped promising field is the interface between topology and the early universe at high energy (although below the Planck scale). UZAN and PETER (1997) showed that if space is multi-connected on scales now smaller than the horizon size, the topological defects such as strings, domains walls,.. expected from GUT to arise at the phase transitions, were very unlikely to form.

In my opinion, a major break through in the field of cosmic topology would be to relate the topological length scale L_T *with cosmological constant A. In an unified scheme with two fundamental lengths scales the Planck scale* l_p *and the inverse square root of the cosmological constant A, a consistent theory of the quantum gravity should be able to predict the most probable value of* L_T *in terms of* l_p *and* $A^{-1/2}$ *. Preliminary calculations in 2-D gravity models can be performed to test the idea.*

Name	Volume	r_-	r_+	$l_{\min}$
WMF	0.9427	0.5192	0.7525	0.5846
Thurston	0.9814	0.5354	0.7485	0.5780
s556(−1,1)	1.0156	0.5276	0.7518	0.8317
m006(−1,2)	1.2637	0.5502	0.8373	0.5750
m188(−1,1)	1.2845	0.5335	0.9002	0.4804
v2030(1,1)	1.3956	0.5483	1.0361	0.3662
m015(4,1)	1.4124	0.5584	0.8941	0.7942
s718(1,1)	2.2726	0.6837	0.9692	0.3392
m120(−6,1)	3.1411	0.7269	1.2252	0.3140
s654(−3,1)	4.0855	0.7834	1.1918	0.3118
v2833(2,3)	5.0629	0.7967	1.3322	0.4860
v3509(4,3)	6.2392	0.9050	1.3013	0.3458

Tab. 1. Small CHMs from SnapPea.

STANDARD COSMOLOGY:

- ***The Friedmann-Lemaitre-Robertson-Walker Model:***

A homogeneous and isotropic universe is one that can be sliced into maximally symmetric 3 spaces of constant curvature and these symmetries constrain greatly the possible allowed solutions to the global metric. Friedmann, Robertson and Walker proposed a metric which gives us all possible solutions for such constant curvature universes. The solutions include big bang solutions, de Sitter solutions and also includes those requiring a cosmological constant.

$The\ Friedmann - Lemaitre - Robertson - Walker$ *metric is of the form:*

$$ds^2 = g_{\mu\nu}dx^\mu dx^\mu = -dt^2 + a^2(\eta)[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + sin^2\theta d\phi^2)]$$

Where $a(t)$ is the scale factor and $\kappa = 0,-1,1$ corresponds to a flat, positively curved or negatively curved universe respectively.

Converting to conformal coordinates where the manifold can be thought of as a constant curvature manifold and the dynamics is incorporated into the conformal scale factor $a(\eta)$ using the definition of conformal time$d\eta = dt/a(t)$. The expression now becomes

$$ds^2 = a^2(\eta)[-d\eta^2 + d\sigma^2]$$

where the spatial part of the metric is given by

$$d\sigma^2 = d_\chi{}^2 + f(x)(d\theta^2 + sin^2\theta d_\phi{}^2$$

$f(\chi)$ is a function of curvature given by

$$f(\chi) = (\chi^2 \ : \ r = \chi \ : \ flat)$$

$$f(\chi) = \left(sinh^2{}_\chi \ : \ r = sinh_\chi \ : hyperbolic\right)$$

$$f(\chi) = (sin^2{}_\chi : \ r = sin_\chi \ : \ spherical)$$

- ***Einstein's General theory of Relativity***

Einstein related the curvature of space-time to matter with the local Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Where $G_{\mu\nu}$ represents a function of the metric of space-time and $T_{\mu\nu}$is the energy momentum tensor representing the matter distribution in space. These equations quantify the influence of matter fields on local curvature, however they do not determine the large-scale topology of the universe.

The Einstein's equation $G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$ is a non-linear system of ten partial differential equations. In the case of a $Friedmann - Lemaitre - Robertson - Walker(FLRW)$ universe, it reduces to two ordinary differential equations which can be rearranged to give us the $Friedmann$ equations below:

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = 8\pi G_{N\rho},$$

$$3\frac{\ddot{a}}{a} = -4\pi G_N(\rho + 3p)$$

From these equations, one can obtain the energy continuity equation given by

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$$

we deduce that the global energy density is a function of curvature. The correspondence is made in terms of the density parameter, $\Omega = \rho/\rho c$ where ρ_c is the total density of a flat universe and can be obtained by taking $\kappa = 0$ which gives us $\rho_c = 3H^2/8\pi G$

in terms of Ω, we get the expression $H^2 a^2(\Omega - 1) = k$, where it becomes evident that $\Omega > 1$ for $\kappa = 1$ and $\Omega < 1$ for $\kappa = -1$. Hence the universe will be positively curved for $\kappa = 1$ and negatively curved for $\kappa = -1$.

In terms of the curvature radius,

$$R_{curv} = \frac{1}{H|\Omega - 1|^{1/2}}$$

Four Dimensional Manifolds of Curvature (Space Time Topology):

One of the topological structure called "Space-Time" and it is widely used in the Einstein's Special Theory of Relativity. And it comes under as Fourth dimension. A four dimensional space is a mathematical extension which is under the concept of 3-dimensional or 3D space. In this dimension Einstein predict the universe is in the shape of "Curvature" where the gravity which is work in that curvature under the space-time dimension.

There are three types of Space-Time:

1) *Manifold Topology*
2) *Path or Zeeman Topology*
3) *Alexandrov Topology*

Geometry of Space-Time:

In Minkowski space-time which is actually similar to the 3-dimensional Euclidean space but the slide difference between these two are distance which is differed with respect to "Time".

In 3D space, the differential distance "d_r" is defined by

$$d_r{}^2 = d_X . d_X$$

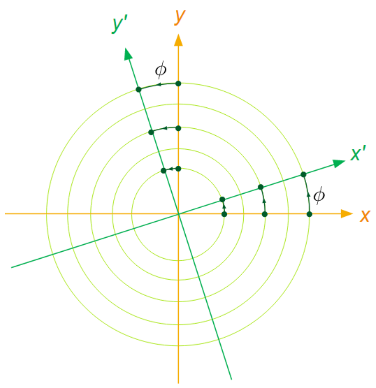
$$= d_{x_1}{}^2 + d_{x_2}{}^2 + d_{x_3}{}^2$$

$$d_x = \left(d_{x_1,} d_{x_2,} d_{x_3}\right)$$

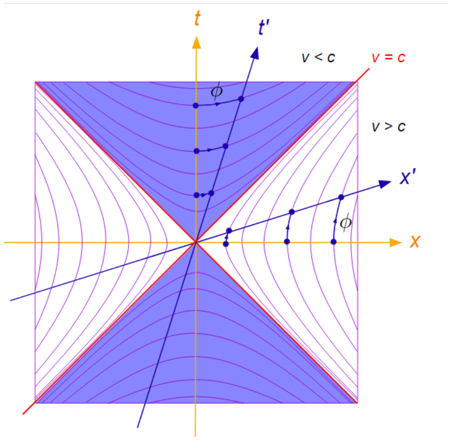
are the differentials of the three spatial dimensions Minkowski geometry, extra dimension with coordinate x^D, which derived from "time". Such that differential distance

$$d_r{}^2 = d_{x_0}{}^2 + d_{x_1}{}^2 + d_{x_2}{}^2 + d_{x_3}{}^2$$

$(d_x = d_{x_0}, d_{x_1}, d_{x_2}, d_{x_3}$) are the differentials of the four space-time dimensions. Special relativity which is simply a Rotational symmetry of the space-time and which is the analogous of the Euclidean space. So the Euclidean space used the Euclidean metric and space-time used the Minkowski metric.

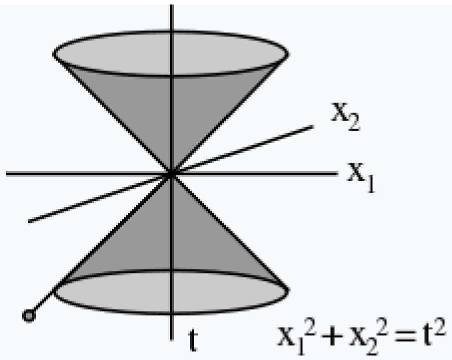


Euclidean space



Minkowski space

- ***3D space-time:***



Three- dimensional dual-cone.

The spatial dimension which is reduced by 2 dimension then it is represent the 3D space

$$d_r^{\,2} = d_{x_1}^{\,2} + d_{x_2}^{\,2} - s^2 d_t^{\,2}$$

The three dimensional dual cone, the null geodesics which is lies along the cone then it becomes

$$\boldsymbol{d_r^{\,2} = 0}$$

$$\boldsymbol{d_{x_1}^{\,2} + d_{x_2}^{\,2} - s^2 d_t^{\,2}}$$

$$\boldsymbol{d_{x_1}^{\,2} + d_{x_2}^{\,2} = s^2 d_t^{\,2}}$$

This is equation of the circle of radius $\boldsymbol{s\, d_t}$.

- ***4D Space-Time:***

The spatial dimension which is extended into 3 dimension the null geodesics are become 4 dimensional cone

Concentric spheres, illustrating in 3-space the null geodesics of a 4-dimensional cone in space-time.

In this concentric circles, the null geodesics in a set of continuous concentric spheres with radii = $s\, d_t$.
This null dual- cone represents the "line of sight" of a point in space. For example when we look at the stars and From the light to I receiving is X years old. And the receiving line is called "The line lying down is the line of sight:

a null geodesics". The looking at an event a distance away and a time $\frac{d}{s}$ *in the past. It is because of the null dual is also called light cone. (In simple words, In the figure the lower point represent the "star", the origin represent the "observer", the line represent the null geodesic is the "the line of sight").*
(The cone in the -t region is the point "receiving").
(The cone in the +t region is the point "sending").

REFERENCE:

1. ATKATZ, D., PAGELS, H., (1982) Phys. Rev. D 25, 2065.
2. BIEBERABACH, L. (1911) "Uber die Bewegungsgruppen der Euklischen Raume" Mathematische Annalen 70, 297. 1912, Ibid, 72, 400.
3. CORNISH, N.J., SPERGEL, D.N., STARKMAAN, G.D. (1996a) phys. Rev. Lett. 77, 215.
4. CORNISH, N.J., SPERAGEL, D.N., STARKMAN, G.D. (1996b) gr- qc/9602039, ubdated version to appear in Class. Q. Grav. (1998).
5. GORNISH, N.H., SPERGEL, D.N., STARKMAN, G.D. (1997) astro-ph /9708225, to appear in phys. Rev. D (May 1998).
6. GORNISH, N.J.,& TUROK, N.G. (1998) Ringing the eigenmodes from compact manifolds", preprint gr-qc 9802066 (to aappear in Class . Quantum Grav.).
7. GORNISH, N.J., GIBBONS, G., WEEKS, J.R. (1998) "Hyperbolic Baby Universes", preprint DAMTP-R98/04.
8. COSTA, S., FAGUNDES, H. (1998) "Birth of a Closed Universe of Negative Spatial Curvature", Preprint gr-qc/9801066.
9. DE OLIVEIRA –COSTA , A., SMOOT, G. (1995) Astrophys. J. 448, 477.
10. DE OLIVERIA- COSTA, A., SMOOT, G., STAROBINSKY, A. (D1996) Astrophys. J. 468, 457.