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EXISTENCE OF NON-CONSTANT CONTINUOUS FUNCTION: URYSOHN'S LEMMA

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Abstract: Urysohn's lemma states that on a normal topological space disjoint closed subsets may be separated by continuous functions in the sense that a continuous function exists which takes value 0 on one of the two subsets and value 1 on the other one (this type of function is called an "Urysohn function"). In fact the existence of such functions is equivalent to a space being normal.

Index Terms - **Urysohn's lemma, dyadic rationals, normal space, Tietze Extension Theorem, Urysohn Metrization Theorem.**

I. Introduction

If $x \neq y$ are two distinct points of a space X, is there a continuous function $f: X \to \mathbb{R}$ such that $f(x) \neq f(y)$? In general this may not be true. There may not exist continuous functions on the given space other than the constants. Let X be a space. Let $\tau = \{\phi, X\}$ be a topology on X. Then $f: X \to \mathbb{R}$ is continuous if and only if f is constant. For each pair of distinct points, if there is an $f \in C(X, \mathbb{R})$ with $f(x) \neq f(y)$, we say that the family $C(X, \mathbb{R})$ separates points. This is the reason for defining the completely regular and normal spaces which ensures plenty of continuous functions. Indeed, the Urysohn's lemma and its corollary, the Tietze extension theorem, provide powerful tools for constructing continuous functions from x to $\mathbb{R}$ with some specified behaviour, for any normal space X. They are among the most widely used tools in topology.

II. Case of Metric spaces

The crucial fact here is the simple observation: If (X, d) is a metric space and $x \in X$, then the function $f_x(y) := d(x, y)$ is continuous on X. For by triangle inequality we have,

$$|f_x(y) - f_x(z)| = |d(x, y) - d(x, z)| \leq d(x, z)$$

Thus $\{f_x : x \in X\}$ is separating family of continuous function on X. More generally, we have

Lemma 1: Let A be any nonempty subset of a metric space (X, d) and $x \in X$. Define $d(x, A) \equiv d_A(x) = \inf\{d(x, a) : a \in A\}$. Then $|d_A(x) - d_A(y)| \leq d(x, y)$ and hence d_A is uniformly continuous on X.

Theorem 1: $d_A(x) = 0$ if and only if x is a limit point of A, where A is any nonempty subset of a metric space (X, d) and $d_A(x)$ defined as above.

Corollary 1: If A is a closed set then $d(x, A) = 0$ if and only if $x \in A$.

Lemma 2: (Urysohn's Lemma for Metric Spaces)

Let A and B be nonempty disjoint closed subsets of a metric space X. Then there exists an $f \in C(X, \mathbb{R})$ such that $0 \leq f(x) \leq 1$ for $x \in X$ and $f = 0$ on A and $f = 1$ on B.

Proof : For any $x \in X$, $d(x, A) + d(x, B) \neq 0$. For, if it were so, then $d(x, A) = 0 = d(x, B)$. Since A and B are closed $x \in A$ and $x \in B$ by the last corollary. This contradicts our hypothesis that $A \cap B = \phi$.

The function $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$ meets our requirments.

Theorem 2: (Tietze Extension Theorem for Metric Spaces)

Let Y be a closed subspace of a metric space (X, d). Let $f: Y \to \mathbb{R}$ be a bounded continuous function. Then there exists a continuous function $g: X \to \mathbb{R}$ such that $g(y) = f(y)$ for all $y \in Y$ and

$$\inf\{g(x) : x \in X\} = \inf\{f(y) : y \in Y\}, \qquad \sup\{g(x) : x \in X\} = \sup\{f(y) : y \in Y\}.$$

Some important things which are used in the next section:

Definition 1: Dyadic Rational

A real number of the form $\frac{m}{2^n}$, where $m, n \in \mathbb{Z}$, is called a dyadic rational or dyadic fraction.

Example 1: $\frac{1}{2}, \frac{3}{8}$ are dyadic rationals but not $\frac{1}{3}$.

Theorem 3: Dyadic Rationals are dense in $\mathbb{R}$.

Proof: Let a and b be two real numbers such that $a < b$. By the Archimedean Property, there exists $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < b - a, \text{whichimplies} \;\; 0 < \frac{1}{2^n} < \frac{1}{n} < b - a.$$

Thus we have $1 < b2^n - a2^n$. As the distance between $b2^n$ and $a2^n$ is greater than 1, there exists $m \in \mathbb{N}$ such that $a2^n < m < b2^n$ which implies that $a < \frac{m}{2^n} < b, (2^n \neq 0)$. So we proved that for each open interval $(a, b) \subset \mathbb{R}$, there exists a rational number of the form $\frac{m}{2^n}$ which belongs to (a, b). In other words, the set of dyadic rationals is dense in $\mathbb{R}$.

III. Case of Topological Spaces

Definition 2: Normal Space

Suppose that one-point sets are closed in X. Then the space X is said to be normal if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

Lemma 3: A space X is normal space if and only if given a closed set F and an open set V containing F there exists an open set U such that $F \subset U \subset \overline{U} \subset V$.

Proof : Let X be normal and F, V as above. Then F and $X\setminus V$ are disjoint closed sets. By normality of X there exist open sets U and W such that $F \subset U$ and $X\setminus V \subset W$ and $U \cap W = \phi$. Since $U \subset X\setminus W$ $\text{isclosed, Weseethat}$ $\overline{U} \subset X\setminus W \subset V$. Thus U is as required. To prove the converse, suppose the condition on the existence of U holds, and let A, B be two disjoint closed subsets of X. Then $A \subset B^c \subset X$ so there exists U open with $A \subset U \subset \overline{U} \subset B^c$. But then $U, \overline{U}^c$ are disjoint open sets with $A \subset U$ and $B \subset \overline{U}^c$. So, X is normal.

Lemma 4: Let X be a normal space. If A and B are disjoint closed subsets of X, for each dyadic rational $r = k2^{-n} \in (0,1]$, there is an open set U_r with the following properties:

(i) $A \subset U_r \subset X\setminus B$,

(ii) $\overline{U}_r \subset U_s$ for $r < s$.

Proof: Let $U_1 = X\setminus B$. Since B is closed, U_1 is open. Again $A \subset U_1$ as A and B are disjoint. Now by the last lemma, there exists disjoint open sets V and W such that

$$A \subset V \subset \overline{V} \subset U_1 \;\text{ and }\; B \subset W.$$

Let $V = U_{\frac{1}{2}}$. Then

$$A \subset U_{\frac{1}{2}} \subset \overline{U}_{\frac{1}{2}} \subset U_1$$

Applying the same lemma once again to the open set $U\frac{1}{2}$ containing A and to the open set U_1 containing $\overline{U}_{\frac{1}{2}}$. We get open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that

$$A \subset U_{\frac{1}{4}} \subset \overline{U}_{\frac{1}{4}} \subset U_{\frac{1}{2}} \subset \overline{U}_{\frac{1}{2}} \subset U_{\frac{3}{4}} \subset \overline{U}_{\frac{3}{4}} \subset U_1$$

Continuing this manner, we construct, for each dyadic rational $r \in (0,1)$, an open set U_r with the following properties:

(i) $\overline{U}_r \subset U_s, 0 < r < s \leq 1$.

(ii) $A \subset U_r, 0 < r \leq 1$.

(iii) $U_r \subset U_1, 0 < r \leq 1$.

More formally, we proceed as follows. We select U_r for $r = k2^{-n}$ by induction on n. Assume that we have chosen U_r for $r = k2^{-n}, 0 < k < 2^n, 1 \leq n \leq N - 1$. To find U_r for $r = (2j + 1)2^{-N}, 0 \leq j < 2^{N-1}$, observe that $\overline{U}_{j2^{1-N}}$ and $X\setminus\overline{U}_{(j+1)2^{1-N}}$ are disjoint closed sets. So once again applying to the last lemma, we can choose an open set U_r such that

$$\overline{U}_{j2^{1-N}} \subset U_r \subset \overline{U}_r \subset U_{(j+1)2^{1-N}}.$$

These U_r's are as desired.

Theorem 4: (Urysohn's Lemma)

Let X be a normal space; let A and B be disjoint closed subsets of X. Then there exists a continuous function

$$f: X \to [0,1]$$

such that $f \equiv 0$ on A and $f \equiv 1$ on B.

Proof: Let r be a dyadic rational in $[0,1]$. Let U_r, s be as in the last lemma. Now we define a function

$$f(x) = \begin{cases} 0 & \text{if } x \in U_r \text{ forall } r, \\ \sup\{r: x \notin U_r\} & \text{otherwise.} \end{cases}$$

Clearly, $0 \leq f \leq 1$. Let $x \in A$. Since $A \subset U_r, 0 < r \leq 1 \Rightarrow x \in U_r, \forall r$. This implies $f(x) = 0$. Since x is arbitrary $f \equiv 0$ on A.
Again let $\in B$. Then for any dyadic rational $r, x \notin U_r$. So, $f(x) = \sup\{r: 0 < r \leq 1\} = 1$. Since x is arbitrary $f \equiv 1$ on B.
Now We need only establish the continuity of f.
Let $x \in X$ be such that $0 < f(x) < 1$. Let $\epsilon > 0$. Choose dyadic rationals s and t in $(0,1)$ such that $f(x) - \epsilon < s < f(x) < t < f(x) + \epsilon$. Then $x \notin U_p$ for dyadic rationals $p \in (s, f(x))$. By the previous lemma $x \notin \overline{U}_s$. On other hand $x \in U_t$. Hence $W = U_t\setminus\overline{U}_s$ is an open

neighbourhood of x.
Now we show that $f(W) \subset (f(x) - \epsilon, f(x) + \epsilon)$.
Let $y \in W$, then from the definition of f we see that $s \leq f(y) \leq t$. In particular $|f(y) - f(x)| < \epsilon$ for $y \in W$. Thus f is continuous.

Corollary 2: Let X be a topological spaces in which all one-point sets are closed. Then X is normal if and only of every pair of disjoint closed sets can be separated by a continuous function in the sense that this continuous function takes value 0 on one of the two subsets and value 1 on the other one(such function is called an "Urysohn function").

IV. Applications

Urysohn's lemma is a key in the proof of many other theorems, for instance:

• **Urysohn Metrization Theorem:** IfX is a normal space with a countable basis (i.e. second countable), then we can use the abundance of continuous functions from X to $[0,1]$ to assign numerical coordinates to the points of X an obtain an embedding of X into $\mathbb{R}^{\omega}$. From this we see that every second-countable normal space is a metric space.

• **Tietze Extension Theorem:** Suppose A is a subset of a space X and $f: A \to [0,1]$ is a continuous function. If X is normal and A is closed in X, then we can find a continuous function g from X to $[0,1]$ that is an extension of f, i.e. $g|_A = f$.

• **Embedding manifolds in $\mathbb{R}^{N}$:** A space X is called a topological n-manifold if each point $x \in X$ has an open neighbourhood $U(x)$ such that U is homeomorphic to an open n-ball.

• **Paracompact Hausdorff spaces equivalently admit subordinate partitions of unity.**

V. Conclusion

Urysohn's lemma is general result that holds in a large class of topologicals spaces. This lemma is the surprising fact that being able to separate closed sets from one another with a continuous function is not stronger than being able to separate them with open sets.

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