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CONVEX SETS IN A TOPOLOGICAL VECTOR SPACE

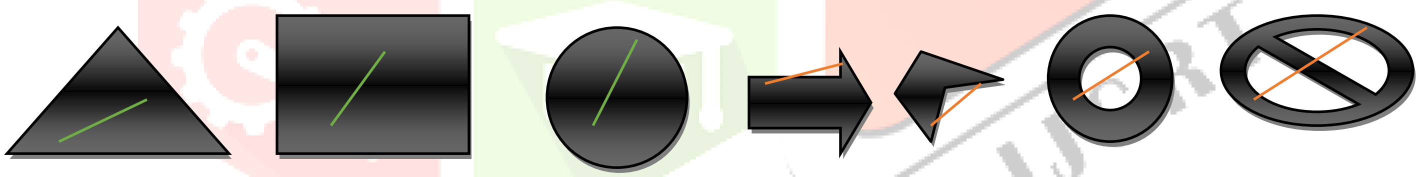
[1]MD JAHID
[1]Assistant Professor(Guest)
[1]Lalit Narayan Mithila University, Darbhanga, Bihar

Abstract

First we will study convex sets and some propositions and theorems related with it in a vector space. When a topology is defined on the vector space in which vector compositions are continuous, then it becomes a topological vector space(TVS). Our aim is to observe the convex behavior of topological terms in TVS.

Introduction

In a general set A, considering its elements as points, said to be a convex set if a line joining any two points of the set A is completely belong to the set A. The name convex is quite similar to the convex surface i.e. a geometrical figure is convex if any part of its edge surface is convex. as..



First three figures are convex whereas last four figures are not convex also in last four figures are collection of points in which a line joining two points of the set does not belong to the set, hence not convex sets.

Definition-1 Let E is a vector space and A is a set in E. The set A is said to be a **convex set** in E if for every pair (x, y) in E $\lambda x+\mu y \in A$ whenever $\lambda \geq 0, \mu \geq 0$ satisfying $\lambda+\mu=1$.

Example-1 The set A={(x,y) : $5x^2 + 3y^2 \leq 9$ } is a convex set in the vector space R^2 .

Solution: let $(x_1, y_1) \& (x_2, y_2)$ are any two points in set A then we can write $5x_1^{\ 2}+3y_1^{\ 2} \leq 9$............(1), $\& 5x_2^{\ 2}+3y_2^{\ 2}$............(2) now for $\lambda \geq 0, \mu \geq 0$ satisfying $\lambda+\mu=1$ i.e. $\mu=1-\lambda$ then consider the point $\lambda(x_1,y_1)+\mu(x_2,y_2)=(\lambda x_1+\mu x_2,\lambda y_1+\mu y_2)=(\lambda x_1+(1-\lambda)x_2,\lambda y_1+(1-\lambda)y_2)$ then

$$\begin{aligned}
&5[\lambda x_1+(1-\lambda)x_2]^2+3[\lambda y_1+(1-\lambda)y_2]^2\\
&=5\lambda^2x_1^{\ 2}+5(1-\lambda)^2x_2^{\ 2}+10\lambda(1-\lambda)x_1x_2+3\lambda^2y_1^{\ 2}+3(1-\lambda)^2y_2^{\ 2}+6\lambda(1-\lambda)y_1y_2\\
&=\lambda^2(5x_1^{\ 2}+3y_1^{\ 2})+(1-\lambda)^2(5x_2^{\ 2}+3y_2^{\ 2})+2\lambda(1-\lambda)(5x_1x_2+3y_1y_2)\\
&\le 9\lambda^2+9(1-\lambda)^2+\lambda(1-\lambda)(10x_1x_2+6y_1y_2)........by(1)\,\&\,(2)\\
&\le 9-18\lambda(1-\lambda)-\lambda(1-\lambda)[5(x_1-x_2)^2+3(y_1+y_2)^2-(5x_1^{\ 2}+3y_1^{\ 2})-(5x_2^{\ 2}+3y_2^{\ 2})]\\
&\le 9-18\lambda(1-\lambda)-\lambda(1-\lambda)[5(x_1-x_2)^2+3(y_1+y_2)^2-18]\\
&\le 9-\lambda(1-\lambda)[5(x_1-x_2)^2+3(y_1+y_2)^2]\le 9\Rightarrow\lambda(x_1,y_1)+\mu(x_2,y_2)\in A
\end{aligned}$$

Which implies that A is a convex set.

Theorem-1 Intersection of convex sets is convex.

Let $S_1\,\&\,S_2$ are convex sets and consider two points in the intersection $S_1\cap S_2$.Then they are contained in the individual sets $S_1\,\&\,S_2$ both . So the line segment connecting them is contained in both the sets $S_1\,\&\,S_2$ and

hence in the set $S_1\cap S_2$. Implies that $S_1\cap S_2$ is a convex set. The same argument can be extended to intersection of finite number of sets and even infinite number of sets too. Since a polytope is an intersection of halfspaces and hyperplanes (linear inequalities and linear equalities), it gives an easier proof that a polytope is convex. But the same property does not hold true for unions. In general, union of two convex sets is not convex. A very simple example is that in the vector space R, intervals [1,2] and [3,4] are convex sets but their union [1,2]U[3,4] is not a convex set.

Definition-2 Let A is a set in vector space E. The **convex hull** of set A, denoted by conv(A) or $\overline{convA}$ is the intersection of all convex sets containing set A. Since intersection of a family of convex sets is convex therefore $\overline{convA}$ is the smallest convex set containing set A. If A is a finite set in vector space E i.e. $A=\{x_i:1\le i\le n\}$ for n is finite, then $\overline{convA}$ is given as the combinations.. $\overline{convA}=\{\sum_{i=1}^{n}\lambda_i x_i:\lambda_i\ge 0,\sum_{i=1}^{n}\lambda_i=1,1\le i\le n\}$.

Definition-3 An **affine set** M in a vector space E is defined as the translation of a subspace S of E by an element **α**∈E, where the subspace S is unique for the affine set M and hence we can say that the subspace S is uniquely determined by M and then we say that S is parallel to M. Thus M is an affine set if M=**α**+S for S being a subspace of E determined by M and for some **α**∈E.

Definition-4 Let A is a set in vector space E then **affine hull** of set A is defined as the intersection of all affine sets containing the set A. Affine hull of a set A is denoted by *aff(A)* or $\overline{aff}\,A$. The intersection of a family of affine sets is an affine set therefore *aff(A)* is a smallest affine set containing set A. Affine hull and convex hull of a set A is related with a relation given by ***aff(A)*** $=$ ***aff(***$\overline{convA}$ ***)*** $=$ ***aff(***$\overline{A}$ ***).***

Definition-5 Let A is a non-empty set in a vector space E and α∈A, then the point α is called a **relative interior point** of convex set A if there exists an open sphere S centered at point α such that S∩*aff(A)*⊂A i.e. α is an interior point of set A relative to *aff(A).* The set of all relative interior points of set A is called **relative interior** of set A and denoted by *ri(A).* The set A is called relative open if relative interior of set A is A itself i.e. *ri(A)* = A.

Theorem-2 When A is convex then ri(A) is non-empty.

Now, when A is convex , then for two points α,β ∈ A then the line joining these two points is completely contained in an open sphere contained in the set A and the lines α+0 and β+0 lie in *aff(A)* , then we observe that the line α+β ∈ S∩*aff(A)* then the set ri(A) is non-empty.

Theorem-3 A subset A of vector space R^n is convex iff convex combination of points of set A is again in set A.

The convex combination of two points $x_1, x_2 \in A$ is written as $\lambda_1 x_1 + \lambda_2 x_2; \lambda_1 + \lambda_2 = 1$ where $\lambda_1, \lambda_2 \geq 0$. when A is convex then $\lambda_1 x_1 + \lambda_2 x_2 \in A$ and after generalizing this concept for n elements we will find the convex combination of n elements of set A is again in set A. Conversely, if convex combination of n elements of set A belongs to the set A then by putting n=2, we find the convex combination of any two elements $x_1, x_2 \in A$, $\lambda_1 x_1 + \lambda_2 x_2 \in A$ then A is a convex set. Which proves the theorem.

Proposition-1 The translation α+A and magnification αA for ∈E of a convex set A in vector space E is a convex set.

Generalizing this concept, we will find that the sum of two convex sets A and B in vector space E defined as $\{\alpha+\beta : \alpha \in A, \beta \in B\}$ is a convex set in E and the product AB=$\{\alpha\beta : \alpha \in A, \beta \in B\}$ is a convex set.

Definition-5 Let E is a vector space over a field K(real or complex) and a topology τ is defined on it. The set E is called a **topological vector space** if the maps (I) $(x, y) \to x + y$ from $E \times E \to E$ and (II) $(\lambda, x) \to \lambda.x$ from $K \times E \to E$ are continuous and then it is abbreviated by TVS. The topology defined on $E \times E$ is the product

topology $\tau \times \tau$ and the topology defined on $K \times E$ is the product topology $\mu \times \tau$ where μ is the usual topology defined on the field K.

Proposition-2 If E is a TVS , then for convex subset A of E, if $x \in \overset{0}{A} \,\&\, y \in \overline{A}$ then the open line segment joining $x \,\&\, y$ is interior to A. i.e. $x \in \overset{0}{A} \,\&\, y \in \overline{A}$ then for a scalar $0 < \lambda < 1$, $\lambda x + (1-\lambda) y \in \overset{0}{A}$.

Proposition-3 If A is a convex set in a TVS E such that $\overset{0}{A} \neq \phi$, then the closure $\overline{A}$ of A is equal to the closure of $\overset{0}{A}$ and the interior $\overset{0}{A}$ of A is equal to the interior of $\overline{A}$ i.e. $\overline{A} = \overline{(\overset{0}{A})}$ and $\overset{0}{A} = (\overset{0}{\overline{A}})$.

Theorem-4 Let E is a TVS and A is a convex set in E, then $\overset{o}{A} \,\&\, \overline{A}$ are also convex sets in E.

<u>Proof:</u> We know that $\overset{0}{A} \subset \overline{A}$ therefore, for any $x, y \in \overset{0}{A} \Rightarrow x \in \overset{0}{A} \,\&\, y \in \overset{0}{A}$ which implies that $x \in \overset{0}{A} \,\&\, y \in \overline{A}$, then by using proposition-2, for a scalar $0 < \lambda < 1$, $\lambda x + (1-\lambda) y \in \overset{0}{A}$ which implies that $\overset{0}{A}$ is convex.

Now to show that $\overline{A}$ is convex, let $x, y \in \overline{A}$ and α,β>0 such that α+β=1. Let N be a neighborhood of $\alpha x + \beta y$. By the definition-5, the mapping (u,v)→αu+βv is continuous from E×E→E, therefore, there exist a neighborhood U of x and a neighborhood V of y such that $\alpha U + \beta V \subset N$. Now, since U∩A≠ϕ and V∩A≠ϕ therefore we can have a∈U∩A and b∈V∩A such that αa+βb∈N∩A⇒ αa+βb∈ $\overline{A}$. since x ∈U∩A and y ∈V∩A therefore $\alpha x + \beta y \in \overline{A}$ which implies that $\overline{A}$ is convex.

<u>Result</u>

Proposition-2, proposition-3 and theorem-4 clears that the behavior of topological terms like interior and closure of a set A changes when A is convex.

Keywords

Affine, Closure, Combination, Hull, Interior, Relative, Translation etc.

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