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STUDY OF SEQUENCES IN A TOPOLOGICAL SPACE

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Abstract

We study sequences in a topological spaces , their formation and convergence. Nets and filters are new devices like sequences, which are constructed with the help of a directed set and all these three are equivalent to each other. In this article, we will find the techniques 'how a sequence can be defined in terms of nets and filters.

Keywords

Co-finite, Eventually, Hausdroff, Limit, Metric, Mapping, Metrizable, Neighborhood, Partially ordered relation etc.

Introduction

A sequence, in a non-empty set X, is universally defined as a mapping from the set of natural numbers N to the set X, therefore this concept is also valid if X being a topological space but the concept of convergence changes. In a non-empty set X in R , a convergent sequence has unique limit , but in a topological space , it is not so. Since convergence is a basic property of a sequence which can arise many differences in properties of the set X and a topological space, hence we require to have an analogous theory of convergence for an arbitrary topological space. Furthermore, topology of a space can be described completely in terms of convergence of sequences. In 1922, E.H.Moore and Herman L.Smith introduces the concept of nets to generalize the notion of a sequence. A directed set, equipped with the relation '≦' or '≧' makes the main role in studying sequence in a topological space, through two new structures *nets* and *filters.* Differ from the concept in general sets, in which a convergent sequence has a unique limit but in a topological space a convergent sequence or a convergent net or a convergent filter has one limit or more than one limit, but on addition a property like Hausdorff or metric in topological spaces, all these three have unique limit. This article covers the concept of sequences, nets and filters and analyse their convergence and equivalency. This article introduces the concept of sequences independently and via nets and filters.

Sequences in a topological space

Let X be a topological space. A sequence in X is defined as a mapping from the set of natural numbers N to X. The general term of a sequence is denoted by x_n and the sequence is denoted by (x_n). If A be a set in the topological space X such that x_n ∈A, ∀ n∈N, then the sequence (x_n) is in the set A, and if there exists a positive integer 'm' such that x_n ∈A, n∈N where n≥m then it is called that (x_n) is *eventually* in the set A.

A topological space can be understood via two ways, general topology and metric topology since in a metric space, a topology must be defined but converse is not true. A question arises, what happens if the set X is finite, in this case, a sequenc (x_n) has infinite number of terms 'constant' . An example of such a sequence is (x_n) = {a,k,b,c,k,d,k,k,k,k,e,k,k,k,……………..}

Convergence of a sequence

A sequence (x_n) in a topological space X is said to converge at a point x∈X iff for every neighbourhood G of x there exists a positive integer 'm' such that x_n ∈G for every n≥m, then we say that x is a limit of sequence (x_n) and written as x_n →x for n→∞ .

Difference of convergency of a sequence in a set in R and in a topological space

Sequence in a non empty set A of R and in a topological space X can be distinguished as 'a sequence in A, if convergent, converges to a unique limit' but in a topological space X, a convergent sequence may have more than one limit. It can be understood by this example

Example-1 In co-finite topology T, on the real line R, let (x_n) be a sequence in R with distinct terms.For any real number p∈R, let G be any open set in R containing point p .Now by the definition of co-finite topology, the complement of G i.e. G^c is a finite set in R and hence containing finite number of elements of the given sequence (x_n) and thus the open set G contains all elements of the given sequence except finite number of elements and hence p is a limit of the given sequence. Since p is arbitrary, so the sequence (x_n) converges to any real number p∈R. *Thus we observe that in a topological space , a sequence may converge at more than one point.*

In a topological space, a convergent sequence has unique limit iff it is Hausdorff space and. A metric space is Hausdorff space and also a topology can be defined on every metric space, therefore a metric space is also called a metric topology and if a metric can be defined on a topological space, it is called metrizable space. In these two kinds of topological spaces, a convergent sequence converges to a unique limit.

In a general topological space X, an accumulation point of a set A in X need not be a limit of a sequence in the set A.

The concept of sequence can also be understood via two different terms nets and filters as ……………………………

Concept via Net

The concept of net introduced by E.H.Moore and Herman L.Smith in 1922 to generalize the notion of a sequence. To define net, let us first define a *directed set.* A partially ordered set M, with the relation '≦' is called a ***directed set*** when for any two elements x,y∈M , there always exists z∈M for which x≦z and y≦z . The set M is called ***inversely directed*** if for any pair of elements x,y∈M, there always exists z∈M such that x≧z and y≧z and a *totally ordered* set is both directed and inversely directed.

A ***net*** in a set X is a mapping whose domain is a directed set. If (D,≦) be a directed set as domain and S= (α_n) for n∈D, be the range of mapping , then , the net is written by (S,≦). A net is thus, sometimes called a directed set. Since N is also a directed set, if the directed set D is replaced by the set of natural numbers N, then, the net thus obtained, gives the general definition of a sequence. If A be a set in a topological space X such that α_n ∈A, ∀ n∈D , then the net (S,≦) is called is **in set A** and if there is some integer m∈D such that α_n ∈A for every n≥m then we say that the net (S,≦) is *eventually* in set A and called frequently in set A iff for each m∈D, there exists n∈D such that n≥m, α_n ∈A. A net (S,≦) in a topological space (X,T) converges to a point a∈X iff it is **eventually** in each neighbourhood N of a. The notion of convergence of a net (S,≦) in a topological space (X,T) depends on the function S , topology T and the relation '≦' . A net may have no limit, unique limit or multiple limit.

Independently, a net (S,≦) in a topological space (X,T) said to **converges** to a point a∈X iff for each neighborhood U of a , there is some m∈D , such that α_n ∈U for every n≧m. A net in a topological space (X,T) converges to a unique limit if if X is Hausdorff space and hence, in a metric space, a convergent net has unique limit. A net A has x∈X as cluster point iff it has a subnet which converges to x. Thus we observe that net and sequence in a topological space (X,T) are equivalent, therefore we can also define each terms like accumulation point, closure of a set, boundary …..etc in terms of net as well.

Concept via filter

A collection F of subsets of a nonempty set X is called a **filte*r*** on set X if it satisfies the following conditions……

1) If A⊂X and B∈F such that A⊃B then A∈F.
2) The intersection of finite collection of sets in class F belongs to F . and
3) The empty set ϕ does not belong to F.

If (X,T) is a topological space and x∈X, then the collection B(x) of all neighborhoods of x forms a filter on space X.

Now let (S,≦) be a net in topological space (X,T) , where S= { α_n : n∈D} and (D,≦) is a directed set . Let F(S) be a family of all subsets A of X with the property that there exists a m∈D such that α_n ∈A whenever m≧n , then it is clear that F(S) is a filter in the space X.If the net (S,≦) is convergent and lim.S = a, then F(S) is also convergent and converges to the same point a i.e. lim.F(S) = lim.S = a .

In a topological space X, a filter F converges at a point x∈X, is called limit of filter F if every neighborhood of x belongs to F, and then we write it as lim.F = x. A filter has also no limit, unique limit or multiple limitbut if X is Hausdorff space then it has unique limit. Conversely, let F be a filter in a topological space (X,T) and D be the set of all pairs (x,A) where x∈A∈F and let us define that (x_1,A_1) ≦ (x_2,A_2) if A_2 ⊂A_1 , then the set D is directed by the relation '≦' and the set S={x_n : n∈D} be a net .

Result

Thus we observe that all the three devices; sequence , net and filter are equivalent and the directed set is common, which construct them and also analyse their convergence .

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