IJCRT.ORG ISSN : 2320-2882
INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

APPLICATION OF THE GENERALIZED DIFFERENCE OPERATOR OF THE SECOND KIND

[1]A.Abisha, [2]Dr. G. Dominic babu
[1]M.phil Scholar, [2]Assistant professor
[1]Manonmanium sundaranar university,
[2]Manonmanium sundaranar university

Abstract:

In this paper, the authors extend the theory of the generalized difference operator Δ_l to the Generalized difference operator of the second kind $L = \{l_1, l_2\}$ for the positive reals l_1, l_2 and obtain some interesting results on second kind . Also by defining its inverse, we establish a few formulae for the sum of third partial sums of products of several powers of the arithmetic progressions in number theory. Appropriate examples are provided to illustrate the results.

Keywords: Generalized difference operator; Generalized polynomial factorial; inverse operator.

Introduction

The theory of difference equations is based on the operator Δ defined as

$$\Delta u(k) = u(k+1) - u(k), k \in \mathrm{N} = \{0, 1, 2, \ldots\} \quad (1)$$

Even though many authors [1 − 3] have suggested the definition of the difference operator Δ as

$$\Delta_l u(k) = u(k+l) - u(k), l \in (0, \infty) \quad (2)$$

no significant progress took place on this line. But,in 2006 [3] ,by taking the definition of Δ as given in [2], the theory of difference equations was developed in a difference direction and many interesting results were obtained in number theory [4].

The theory was then extended for real $l \in (0, \infty)$ and

$$\Delta_{-l}u(k) = u(n - l) - u(k)$$

and based on this definition they studied the qualitative properties of a particular difference equation and no one else has handled this operator.

In this paper, we develop theory for Δ_L, the generalized difference operator of second kind and obtain some significant results, relations and formule in number theory.

Throughout this paper, we make use of the following assumptions:

1. q is the positive integer and l is a positive real,

2. $[x]$ denotes the integer part of x,

3. $N = \{0, 1, 2, . . .\}$, $N(a) = \{a, a + 1, . . .\}$

4. $L = (l_1, l_2, . . . , l_n)$

5. $0(L) = \{\varphi\}$

6. $p(L) = \{\{l_1, l_2\}\}$

8. $N_l(j) = \{j, j + l, . . .\}$

Preliminaries

In this section, we present some definitions and some results on generalized difference operator and polynomial factorials, which will be useful for subsequent discussion.

Definition 2.1

For any real or complex valued function $u(k)$ on $[0, \infty)$, the generalized difference operator of the second kind, denoted by $\Delta_{(l_1),l_2}$ is defined as

$$\Delta_{(l_1,l_2)}u(k) = u(k + l_1 + l_2) - (u(k + l_1) + (u(k + l_2)) + u(k) \quad (3)$$

Definition 2.2

If t is a positive integer, then generalized polynomial factorial of the second kind is defined by

$k^{n}_{l_1,l_2} = (k + l_1)^{n}_{l_2} + (k + l_1)^{n}_{l_2}$-$(k^{n}_{l_2} + k^{n}_{l_1})$ (4)

Where $k_l^{(t)} = k(k - l)(k - 2l) \dots (k - (t - 1)l)$.

Definition 2.3

The inverse of the forward difference operator of the second kind denoted by $\Delta^{-1}_{l_1,l_2}$ is defined as

$\Delta^{-1}_{l_1,l_2}u(k)$=$\Delta^{-1}_{l_2}(\Delta^{-1}_{l_1}u(k))$ (5)

3.Main results

In this section, we derive the formula for the sum of general partial sums of products of several powers of consecutive terms of an arithmetic progression.

Lemma 3.1

If E is the usual shift operator, then

1. $\Delta_{(l_1,l_2)} = E^{(l_1+l_2)} - (E^{(l_1)} + E^{(l_2)}) + 1$

2. $\Delta_{(l_1,l_2)} = \Delta_{(l_1+,l_2)} - \Delta_{(l_1)} + \Delta_{(l_2)}$

3. $\Delta_{l_1,l_2} = \Delta_{l_1}\Delta_{l_2}$

4. If l_j's are positive integers, then

$$\Delta_{l_1,l_2} = \left(\sum_{i=1}^{l_1} l_1\, C_i \Delta^{j}\right)\left(\sum_{j=1}^{l_2} l_2\, C_j \Delta^{j}\right)$$

Lemma 3.2

If s^n_r 's are the stirling numbers of the first kind, then

$\sum_{r=1}^{n} s^n_r\, t^{n-r} \Delta_{l_1,l_2} k^r = \Delta_{l_1,l_2} k_1^{(n)}$. (6)

Lemma 3.3

$$Let\ \ m\ be\ a\ positive\ integer\ \ and\ (n-1)(L) = \{\{l_1, l_2, \dots l_{n-1}\},$$
$$\{l_1, l_2, \dots l_{n-2}, l_n\}, \quad \dots, \{l_2, l_3, \dots l_n\}\}, then$$

$$\Delta_{l_1,l_2} \boldsymbol{k}_{l_1,l_2}^{(n)} = \boldsymbol{nl_2} \Delta_{l_1,l_2} \boldsymbol{k}_{l_2}^{(n-1)} + \boldsymbol{nl_1} \Delta_{l_1,l_2} \boldsymbol{k}_{l_1}^{(n-1)}$$

Theorem 3.4

If l is positive real and j=k-$\left[\frac{k}{l}\right]$l,then

$$\Delta_l^{-1} u(k) = \sum_{r=1}^{\left[\frac{k}{l}\right]} u(\text{k-rl}) + c_{j,} \qquad (7)$$

Where c_j is constant for all k∈ N_l(j).

Theorem 3.5

Let k∈ $[2l_{2,}\infty), l_2 \geq l_{1,}$ u_2 (k)=$\Delta_{l_2}^{-1}$ (u_1(k)- $u_1(j_1)$) and

u_1 (k)=$\Delta_{l_1}^{-1} u(k)$ with $j_i =$ k – $\left[\frac{k}{l_i}\right] l_i$, for i=1,2. Then

$$\sum_{r_2=1,}^{r_2^*} \sum_{i=1}^{r_1^*} u(k - r_2 l_{2-} r_1 l_1) = u_2(k) - u_2(j_2) - U_{12}, \qquad (8)$$

Where $r_{2=}^* \left[\frac{k}{l_2}\right], r_1^* = \left[\frac{k - r_2 l_2}{l_1}\right], u_2(k) = \Delta_{l_1}^{-1}\left(\Delta_{l_2}^{-1} u(k)\right)$

$j_{1r_2} = (k - r_2 l_2) - \left[\frac{k - r_2 l_2}{l_1}\right] l_1$ and $U_{12} = \sum_{r_2=1}^{\left[\frac{k}{l_2}\right]} u_1(j_{1r_2})$

Proof:

From the above equation and replacing k by j_1, we get

$$z_1(k) = \Delta_{l_1}^{-1} u(k)$$

$$\sum_{r_2=1}^{\left[\frac{k}{l_1}\right]} u(k - r_1 l_1) = u_1(k) -, u_1(j_1),$$

Where $u_1(k) = \Delta_{l_1}^{-1} u(k)$.

The proof follows by taking $\Delta_{l_2}^{-1}$ on both sides and substituting k by j_2

$$\sum_{r_2=1,}^{r_2^*} \sum_{i=1}^{r_1^*} u(k - r_2 l_{2-} r_1 l_1) = u_2(k) - u_2(j_2) - U_{12}$$

Theorem 3.6

Let $l \in (0, \infty)$, and $u(k)$ be real valued function defined on $[0, \infty]$. If $k \in [3l, \infty)$, then

$$\sum_{r=2}^{\frac{k}{l}} \binom{r-1}{m-1} u(k-rl) = u_2(k) - u_2(j) \qquad (9)$$

Where u_2 (k)=$\Delta_{l_2}^{-1}$ (u_1(k)- $u_1(j_1)$) and

u_1 (k)=$\Delta_{l_1}^{-1} u(k)$ with $j_i =$ k – $\left[\frac{k}{l_i}\right] l_i$

Proof

From (8) and replacing k by j, we get

$$\sum_{r=1}^{\left[\frac{k}{l_1}\right]} u(k-rl) = \Delta_{l_1}^{-1} u(k) -= \Delta_{l_1}^{-1} u(j)$$

Taking $u_1(k) = \Delta^{-1} u(k)$, (9) becomes ,

$\sum_{r=1}^{\left[\frac{k}{l_1}\right]} u(k-rl) = u_1(k) - u_1(j)$

follows from (7) and (8).

Corollary 3.7

If k$\in [2l_{2,}\infty), l_2 \geq l_{1,}$ Then

$$\sum_{r_2=1,}^{r_2^*} \sum_{i=1}^{r_1^*} u(k - r_2 l_{2-} r_1 l_1) a^{k - r_2 l_{2-} r_1 l_1} = u_2(k) - u_2(j_2) - U_{12}, \qquad (10)$$

Where $r_{2=}^* \left[\frac{k}{l_2}\right], r_1^* = \left[\frac{k - r_2 l_2}{l_1}\right]$

Where $u_2(k) = \frac{1}{a^{l_1}-1} \left(\frac{a^k}{a^{l_2}-1} \left(k - \frac{l_{2a^{l_2,}}}{a^{l_2}-1} \right) \right) - \frac{l_{1a^{l_1,}}}{a^{l_1}-1^2} \left(\frac{a^k}{a^{l_2}-1} \right)$

$$- \frac{a^j}{a^{l_1}-1} \left((j_1) - \frac{l_{1a^{l_1,}}}{a^{l_1}-1^2} \right) \frac{k_{l_2}{}^{(1)}}{l_2},$$

is obtained from

$$u_1(k) = \frac{1}{a^{l_1}-1} \left(k a^k - j_1 a^{j_1} \right) - \frac{l_{1a^{l_1,}}}{a^{l_1}-1^2} \left(a^k - a^{j_1} \right)$$

$j_{1r_2} = (k - r_2 l_2) - \left[\frac{k - r_2 l_2}{l_1}\right] l_1$ and $U_{12} = \sum_{r_2=1}^{\left[\frac{k}{l_2}\right]} u_1(j_{1r_2})$

Theorem 3.8

Let $l = \sum_{i=1}^{n} l_i$, $k \in [l, \infty)$ and $l_2 = m(l_1)$, $(multiple\ of\ (l_1))$

Then

$$\sum_{r_2=1,}^{r_2^*} \sum_{i=1}^{r_1^*} u(k - r_2 l_2 - r_1 l_1) = u_2(k) - u_2(j_2), \qquad (11)$$

Where $r_2^* = \left[\frac{k}{l_2}\right], r_1^* = \left[\frac{k - r_2 l_2}{l_1}\right]$, u_2 (k)=$\Delta_{l_2}^{-1}$ (u_1(k)- $u_1(j_1)$) and

u_1 (k)=$\Delta_{l_1}^{-1} u(k)$

Corollary 3.9

Let u(k)= k^m, $k \in [l, \infty)$ and $l = \sum_{i=1}^{n} l_i$, Then

$$\sum_{r_2=1,}^{r_2^*} \sum_{i=1}^{r_1^*} u(k - r_2 l_2 - r_1 l_1)^m = u_2(k) - u_2(j_2), \qquad (12)$$

Where $r_2^* = \left[\frac{k}{l_2}\right], r_1^* = \left[\frac{k - r_2 l_2}{l_1}\right]$, u_2 (k)=$\Delta_{l_2}^{-1}$ (u_1(k)- $u_1(j_1)$) and

u_1 (k)=$\Delta_{l_1}^{-1} u(k)$

4.Applications

In this section, we present some examples to illustrate the main results

Example 2.1

Taking $u(k) = k$ in (9) we obtain

$$\sum_{r_2=1,}^{\left[\frac{k}{l_2}\right]} \sum_{i=1}^{\left[\frac{k - r_2 l_2}{l_1}\right]} (k - r_2 l_2 - r_1 l_1) = u_2(k) - u_2(j_2) - U_{12}$$

Where $u_2(k) = \frac{k_{l_2}^{(3)}}{6 l_1 l_2} + \frac{(l_2 - l_1) k_{l_2}^{(2)}}{4 l_1 l_2} - \frac{(j_1)_{l_1}^{(2)} k_{l_2}^{(1)}}{2 l_1 l_2}$ is obtained from

$u_1(k) = \frac{k_{l_2}^{(2)}}{2l_1} - \frac{(j_1)_{l_1}^{(2)}}{2l_1}$, $U_{12} = \sum_{r_2=1,}^{\left[\frac{k}{l_2}\right]} u_1(j_{1r_2})$ and

$$j_{1r_2} = (k - r_2 l_2) - \left[\frac{k - r_2 l_2}{l_1}\right] l_1.$$

In particular when $k = 10, l_1 = 2, l_2 = 3$ and $j_1 = 1, j_2 = 1$ in (7), we get

$$\sum_{r_2=1,}^{\left[\frac{10}{3}\right]} \sum_{i=1}^{\left[\frac{10-3r_2}{2}\right]} (10 - 3\,r_2 - 2r_1) = u_2(10) - u_2(1) - U_{12}$$

Where

$$u_2(10) = \frac{10_3^{(3)}}{6(2)(3)} + \frac{(3-2)10_3^{(2)}}{4(2)(3)} - \frac{(1)_2^{(2)}10_2^{(1)}}{2(2)(3)}$$

$$u_2(10) = \frac{415}{36}$$

$$u_2(1) = \frac{1_3^{(3)}}{6(2)(3)} + \frac{(3-2)1_3^{(2)}}{4(2)(3)} - \frac{(1)_2^{(2)}1_2^{(1)}}{2(2)(3)}$$

$$u_2(1) = \frac{10}{36}$$

$$U_{12} = \sum_{r_2=1,}^{\left[\frac{k}{l_2}\right]} u_1(j_{1r_2}) = \frac{1}{2}$$

$u_1(k) = \frac{k_{l_2}^{(2)}}{2l_1} - \frac{(j_1)_{l_1}^{(2)}}{2l_1}$, putting k=10

$$\sum_{r_2=1,}^{\left[\frac{10}{3}\right]} \sum_{i=1}^{\left[\frac{10-3r_2}{2}\right]} (10 - 3\,r_2 - 2r_1) = \frac{415}{36} + \frac{10}{36} + \frac{1}{2} = 11$$

LHS = RHS

Example 4.2

Taking $k = 9, l_1 = 2, l_2 = 3, a = 2, j_1 = 1, j_2 = 0$ in (10) we find

$$\sum_{r_2=1,}^{r_2^*} \sum_{i=1}^{r_1^*} u(k - r_2 l_2 - r_1 l_1) a^{k - r_2 l_2 - r_1 l_1} = u_2(k) - u_2(j_2) - U_{12},$$

Proof

$$\sum_{r_2=1,}^{\left[\frac{9}{3}\right]} \sum_{i=1}^{\left[\frac{9-3r_2}{2}\right]} (9 - 3\,r_2 - 2r_1) 2^{9-3r_2-2r_1} = u_2(9) - u_2(0) - U_{12},$$

$$u_2(9) = \frac{1}{2^2-1}\left(\frac{2^9}{2^3-1}\left(9 - \frac{3(2^3)}{2^3-1}\right)\right) - \frac{2(2^2)}{2^2-1^2}\left(\frac{2^9}{2^3-1}\right)$$

$$-\frac{2^1}{2^2-1}\left((1) - \frac{2(2)^2}{2^2-1^2}\right)\frac{9_3{}^{(1)}}{3}$$

$$= \frac{32702}{441}$$

$$u_2(3) = \frac{1}{2^2-1}\left(\frac{2^3}{2^3-1}\left(3 - \frac{3(2^3)}{2^3-1}\right)\right) - \frac{2(2^2)}{2^2-1^2}\left(\frac{2^3}{2^3-1}\right)$$

$$-\frac{2^1}{2^2-1}\left((1) - \frac{2(2)^2}{2^2-1^2}\right)\frac{3_3{}^{(1)}}{3}$$

$$= -\frac{128}{441}$$

$$U_{12} = \sum_{r_2=1,}^{\left[\frac{k}{l_2}\right]} u_1(j_{1r_2}) = \frac{4}{9}$$

$$\sum_{r_2=1,}^{\left[\frac{9}{3}\right]} \sum_{i=1}^{\left[\frac{9-3r_2}{2}\right]} (9 - 3\,r_2 - 2r_1)2^{9-3r_2-2r_1} = 74$$

LHS = RHS

Example 4.3

Taking $k = 2100, l_1 = 3, n = 1, a = 2, j_1 = 0, m = 1$ in (12) we find

$$\sum_{r_2=1,}^{\left[\frac{k}{l_2}\right]} \sum_{i=1}^{\left[\frac{k-r_2l_2}{l_1}\right]} (k - r_2l_{2-}r_1l_1)^m = u_2(k) - u_2(j_2)$$

$$\sum_{r_2=1}^{\frac{2100}{3}}(2100 - 3r_1) = \frac{2100_3{}^{(1)}}{2(3)}$$

$$\sum_{r_2=1}^{\frac{2100}{3}}(2100 - 3r_1) = 733950$$

$$\frac{2100_3{}^{(1)}}{2(3)} = 733950$$

LHS = RHS

5 References

[1] R. P Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.

[2] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Theory of generalized difference operator and its applications, Far East J. Math. Sci. 20(2) (2006), 163171.

[3] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Qualitative properties of solutions of certain class of difference equations, Far East J. Math. Sci. 13 23 (3) (2006), 295304.

[4] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Generalized Bernoulli polynomials through weighted pochhammer symbols, Far East J. Appl. Math. 26 (3) (2007), 321333.

[5] M. Maria Susai Manuel, A. George Maria Selvam, G. Britto Antony Xavier, Rotatory and boundedness of solutions of certain class of difference equations, Int. J. Pure Appl. Math. 33(3) (2006), 333343.

[6] M. Maria Susai Manuel, G. Britto Antony Xavier, Recessive, dominant and spiral behaviours of solutions of certain class of generalized difference equations, Int. J. Differ. Equ. Appl. 10(4) (2007), 423433.

[7] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, Generalized difference operator of the nth kind and its application to number theory, Int. J. Pure Appl. Math. 47(1) (2008), 127140.

[8] M. Maria Susai Manuel, G. Britto Antony Xavier, V. Chandrasekar, R. Pugalarasu, S. Elizabeth, On generalized difference operator of third kind and its applications in number theory, Int. J. Pure Appl. Math. 53(1), (2009), 6981.

[9] R. E. Mickens, Difference Equations, Van Nostrand Reinhold Company, New York, 1990.

[10] S. N. Elaydi, An Introduction to Difference Equations, 2nd ed., Springer, 199