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Some New Results on linear continuous Operators

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Abstract

In this paper we shall prove If E and F are topological vector spaces,then a linear map K of E into F is called compact, if there exist a neighbourhood U of origin n E,whose image set K(U) is relatively compact. Here we find the notion of completely continuous operator to the topological vector space and also the Normed spaces derive a property equivalent to complete continuity.

Keywords: Completely continuous, Zero neighbourhood,

Introduction

A subset M of a topological space R is called compact, if every open covering of M contains a finite sub covering. A subset of a topological space R is called relatively compact if M is contained in a compact subset of R .A subset M of a topological vector space is called bounded if corresponding to every zero neighbourhood U there exists a $\alpha > 0$ such that $\alpha U \supset M$.

Theorem (1.1)

In a Topological Vector space

(a) Every subst of a bounded set is bounded

(b) The continuous image of abounded set is bounded.

(c) The closed envelope of abounded set is bounded

(d) Every compact set M is bounded.

(e) The union of finitely many and the intersection of arbitrary number of bounded sets is bounded.

(f) If M, N are bounded sets then the sets M+N and λM λ∈∅arealso bounded.

The proofs can be found in Kothe's Topological Vector spaces.

Theorem(1.2)

If E and F are normed linear spaces ,K a linear map of E into F and

$$S = \{x \in E : || \quad x || < 1$$

The closed unit ball of E,then the following properties are equivalent

(α) K is completely continuous

(β) K (S) is relatively compact

(γ) K (S) is compact

Here K is completely continuous iff there exists a neighbourhood U of origin in E such that K (U) is relatively compact.

Proof

$(\alpha) \to (\beta)$: If $K \in R\ (E\ ,F\ )\ and\ \{y_n\}$ a sequence fromK(S),then there existsx_n , $n \in N$,with $y_n = Kx_n, x_n \in S$.

$Since$ $\{x_n\}$isa bounded sequence ,there exists a subsequence $\{x_{n_k}\}$ such that that$\{x_{n_k}\}$= $\{Kx_{n_k}\}$ converges to some $y \in\ F$ F then K (S) is relatively compact.

(β) :→ (α):If K (S) is relatively compact and $\{x_n\ \in\ E\ :\ ||\ \ x\ ||\ = \gamma\ \neq 0$a bounded sequence in E the$\gamma^{-1}x_n \in S$n

$K(\gamma^{-1}x_n) \in K(S)$

We can come to the conclusion that there exists a convergent subsequence:therefore $\{Kx_{n_k}\}$ is converges and hence K is completely continuous.

The equivalence of (β) and (γ) are as follows

We now prove the additional proposition.If K isneighbou continuous then U=S goves the desired results. Conversely,ifther exists such a zero neighbourhood in E,then there exist $\rho > 0$with $\supset \rho s = \{x \in E{:}\,||\ x\ || \le \rho\}$. Then also $(\rho s) \in K(U)$; hence

$K(\rho s) = \rho K(S)$lies in a compact set M.$\frac{1}{\rho}.M$i also contact; as

$K(S) \in \frac{1}{\rho}.M, K(S)$is relatively compact and K is completely continuous.

In the following we denote by R(E ,F) the set of all compact maps of a topological vector space E into a topological vector space F.In case of topological vector spaces ,the notion of compact maps is rich in content and more important than the notion of compact maps is rich in content and more important than the notion of completely continuous maps.

Theorem (1.3)

(a) A compact operator is continuous.

(b) Sum and scalar multiple of compact operators are compact.

(c) The product of a compact operator with a continuous operator is compact.

In particular,R(E) is a two sided ideal in the algebra L (E)

Proof

(a) If K ∈ R (E,F) ,then there exists a zero neighbourhoods U in E ,whose image K(U) lies in a compact set M in F .If V is an arbitrary zero neighbourhood in F ,then on account of the boundedness of M there exists $\alpha > 0$ such that $M \subset \alpha V$,therefore$\frac{1}{\alpha}M \subset V$.Hence corresponds to zero neighbourhood V in F ,there exists a zero neighbourhood $\frac{1}{\alpha}U$ in E with $K(\frac{1}{\alpha}U) = \frac{1}{\alpha}K(U) \subset V$ .Therefore ,K is continuous.

(b) If $K_{1,}K_2 \in R(E,F)$, then there exists a zero neighbourhood U_1, U_2 in E and compact sets $M_{1,}M_2$ in F with $K_i(U_i) \subset M_i$,i= 1,2 .Then the intersection $U = U_1 \cap U_2$

Is also a zero neighbourhood in E and it follows that:

$\boldsymbol{K_i(U) \subset M_i, i = 1,2}$ 2 **d**

$$(K_1 + K_2)U = \{(K_1 + K_2)x \ = K_1x + K_2x;\ x \in U\}$$
$$= \{K_1x + K_2y : x, y \in U\}$$
$$= K_1(U) + K_2(U)$$
$$\subset M_1 + M_2$$

But $M_1 + M_2$is compact and hence $K_{1,} + K_2 \in R(E,F)$. In a similar fashion we can prove that the K∈R(E,F) also K∈ $R(E,F)$,F),∈ ∅ .

(c) If$K \in R(E,F)$,$L \in L\ (F,G)$,then thee exists a zero neighbourhood U in E ,whose image K(U) is contained in a compact set M in F .Then also$L(K(U)) \subset L(M)$⊂L(M) L,K is Compact.

If $\alpha \in\ (E\ ,F)$,$K \in R(F,G)$, then there exists a zero neighbourhood V in F ,whose image K (V) lies in a compact set M in G .Since L is continuous $L^{-1}(V) = U$is a zero neighbourhood in E,therefore $K(L(U)) \subset K(V)\ \subset M$,hence $KL \in R(E\ ,G)$.

Theorem (1.4)

Every Finite dimensional continuous map A∈ $\Gamma(E,F)$ is compact.

Proof : As dim A (E) $< \infty$,there exists a compact neighbourhood V of 0 in A(E) ; but the V is also compact subset of F .On account of the continuity of A ,the set$A^{-1}(V) = U\ is$ a neighbourhood of O in E and A(U) =AA^{-1}(V)⊑V i.e A(U) lies in a compact subset V of F ,hence A ∈ R (E ,F)

In the proof of the continuity of a compact operator we have only used that k maps a zero neighbourhood into a bounded set.

Definition (1.1)

A map $T \in L(E,F)$ is called bounded ,if there exists a neighbourhood of O in E ,whose image T(U) is a bounded subset of F.We wish to denote the set of all bounded operators of E into F by ζ (E,F) :

Γ(E,F) ⊂R(E,F) ⊂ζ(E,F)⊂L(E,F)

From above theorem it follows that a finite dimensional continuous operator is compact, a compact operator is bounded and a bounded operator is continuous.

In contrasta continuous operator need not be bounded.Infact if E is locally convex but not normable than I is continuous but not bounded.For if there exists a bounded neighbourhood U with I (U)=U ;then by the theorem of Kolmogarov (e.g P f laumann-Unger| 1| ,p.-121 (e) would be normable.Hence in general ζ(E,F) is aproper subset of L(E,F).In normed linear spaces T is bounded in the sense of definition iff T is bounded in the classical in the classical sense,hence in normed linear spaces ζ(E,F)=L(E,F)

Theorem (1.5)

(a) Sums and scalar multiples of bounded operators are bounded

(b) The product of abounded operator with a continuous operator is bounded.

(c) In particular,(E) is a two sided ideal in the algebra L(E)

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