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GENERALIZED JORDAN DERIVATIONS OF PRIME RINGS

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Abstract. Let A be any prime ring of Ch 6= 2 and $f : A \to A$ be the generalized Jordan Derivation, then we have

(1) $f(a + b) = f(a) + f(b)$

(2) $f(ab) = f(b)a + bd(a)\ \forall\ a,b \in A$ where d is defined as reverse derivation of A.

In this paper, it is shown that

(i) $f(aba) = f(a)ba + ad(b)a + abd(a)\ \forall\ a,b \in A$

(ii) $a^{a+c} = a^b + a^c$ & $a^b = -b^a$ where

$$a^v = f(ab) - f(a)b - ad(b).$$

We proved Herstein [3] Lemma 3.1 P.1106, Havala [2] Def. P.1147 as Corollaries along with other results.

Introduction

We define the generalized Jordan derivations of a ring. Let A be any ring of Ch 6= 2. A mapping $f : A \to A$ is said to be Generalized Jordan derivation if

$$f(a + b) = f(a) + f(b)$$

$$f(ab) = f(b)a + bd(a) \quad \forall\ a,b \in A$$

where d is defined as reverse derivation on A. Our aim is to show that Generalized Jordan Derivation of Prime rings of Ch 6= 2 coincident with Generalized derivations.

In Theorem 3.1, we have proved that if f is generalized Jordan Derivations of A then $f(aba) = f(a)ba + ad(b)a + abd(a)\ \forall\ a,b \in A$.

Key words and phrases. Generalized Jordan derivation, Prime ring, Integral Domain.

1

Replcing f by d we get Herstein [3] Lemma 3.1 P.1106. In Theorem 4.1, we have proved that if $a^b = f(ab) - f(a)b - ad(b)$. Then

$$a_{b+c} = a_b + a_c$$

$$a^b = -b^a\ \forall\ a,b \in A.$$

Using above results, we have proved very important Theorem 5.1. If A is Prime ring of Ch 6= 2 then any generalized Jordan Derivation is generalized derivation of A i.e. $f(ab) = f(a)b + ad(b)$ which is definition of Generalized derivation given by Havala [2] Def. P.1147. Also we take prime ring of $Ch = 2$.

1. Generalized Jordan Derivations

In this section, we study the Generalized Jordan Derivations in a ring. Let A be any ring of Ch 6= 2. A mapping $f : A \to A$ is said to
be Generalized Jordan Derivation if

$$f(a + b) = f(a) + f(b)\ f(ab) = f(b)a + bd(a)\ \forall\ a,b \in A$$

where d is defined as reverse derivation on A.

Our aims is to show that Generalized Jordan derivation for the prime rings coincident with Generalized derivations.

2

In this section, we take A be any prime ring of Ch 6= 2

Definition 2.1. (Prime Ring). Let A be any ring. Then A is said to be Prime Ring iff

$$xay = 0\ \forall\ a \in A$$
$$\Rightarrow x = 0 \quad \text{or} \quad y = 0$$

Remark. We will use the following two Lemma for drawing Theorem 3.1.

Lemma 2.2. *Let A be any ring. For $a \in A$, let*

$$T(a) = \{r \in A : r(ax - xa) = 0\ \forall\ x \in A\}.$$

Then $T(a)$ is two sided Ideal of A.

Lemma 2.3. *If A is prime ring and if $a \in A$ is not in Z, centre of A, Then*

$$T(a) = 0.$$

3. Generalized Jordan Derivations of Prime rings

Theorem 3.1. *If f is generalized Jordan Derivations of A then $f(aba) = f(a)ba + ad(b)a + abd(a)\ \forall\ a,b \in A$.*

Proof. Since $f(a^2) = f(a)a + ad(a)$. Putting $a = a + b$

$$\begin{aligned}\Rightarrow f(a+b)^2 &= f(a+b)(a+b) + (a+b)d(a+b)\\ &= (f(a)+f(b))(a+b) + (a+b)(d(a)+d(b))\\ &= f(a)a + f(b)a + f(a)b + f(b)b + ad(a)\\ &\quad +bd(a) + ad(b) + bd(b). \qquad (1)\end{aligned}$$

Also

$$\begin{aligned}f(a+b)^2 &= f(a^2 + b^2 + ab + ba)\\ &= f(a^2) + f(b^2) + f(ab + ba)\\ &= (f(a)a + ad(a)) + (f(b)b + bd(b)) + f(ab + ba)\ (2)\end{aligned}$$

■

From (1) and (2) we have $f(ab + ba) = f(a)b + ad(b) + f(b)a + bd(a)$ (3)

Consider $E = f(a(ab + ba) + (ab + ba)a)$. Now

$$E = f(a)(ab + ba) + ad(ab + ba) + f(ab + ba)a + (ab + ba)d(a)$$
$$= f(a)ab + f(a)ba + ad(ab) + ad(ba) + f(ab)a + f(ba)a = +abd(a) + bad(a) \text{ (by using (3))}$$

$$\Rightarrow E = f(a)ab + f(a)ba + a(d(b)a + bd(a)) + a(d(a)b + ad(b)$$
$$= +(f(b)a + bd(a))a + (f(a)b + ad(b))a + abd(a) + bad(a)$$

$$\Rightarrow E = f(a)ab + f(a)ba + ad(b)a + abd(a) + ad(a)b + a^2d(b) + f(b)a^2 + bd(a)a + f(a)ba + ad(b)a + abd(a) + bad(a) \quad (4)$$

On the other hand

$$E = f(a^2b + aba + aba + ba^2)$$
$$= f(a^2b + 2aba + ba^2)$$
$$= f(a^2b + ba^2) + 2f(aba)$$
$$= f(a^2)b + a^2d(b) + f(b)a^2 + bd(a^2) + 2f(aba)$$
$$= (f(a)a + ad(a))b + a^2d(b) + f(b)a^2 + b(d(a)a + ad(a)) + 2f(aba)$$

$$\Rightarrow E = f(a)ab + ad(a)b + a^2d(b) + f(b)a^2 + bd(a)a + bad(a) + 2f(aba) \quad (5)$$

From (4) and (5) we have

$$2f(aba) = f(a)ba + ad(b)a + abd(a) + f(a)ba + ad(b)a + abd(a)$$
$$\Rightarrow 2f(aba) = 2f(a)ba + 2ad(b)a + 2abd(a)$$
$$\Rightarrow f(aba) = f(a)ba + ad(b)a + abd(a)$$

Hence proved.

Remark. Replacing f by d we get Herstein [3] Lemma 3.1 p.1106.

Corollary 3.2. *If f is a generalized Jordan Derivation of A then* $\forall\, a,b,c \in A$

$$f(abc + cba) = f(a)bc + ad(b)c + abd(c) + f(c)ba + cd(b)a + cbd(a)$$

Corollary 3.3. *If* $a,b \in A$ *and if* $ab = 0$. *Then* $\forall\, c \in A$

$$f((ba)c) = f(ba)c + bad(c)$$

Corollary 3.4. *Let* $V = \{a \in A : f(ax) = f(x)a + xd(a)\ \forall\, x \in A\}$.
If $ab = 0$ *then by 3.3*

$$f((ba)c) = f(ba)c + bad(c)$$
$$\Rightarrow b\,a \in V$$

4. More results on Generalized Jordan Derivation

Theorem 4.1. *If* $a^b = f(ab) - f(a)b - ad(b)$. *Then P. T.*

(i) $a_{b+c} = a_b + a_c$

(ii) $a^b = -b^a \quad \forall\, a\, b \in A$

Proof. (i) Now

$$a_b = f(ab) - f(a)b - ad(b)$$

$$a_c = f(ac) - f(a)c - ad(c)$$

Then

$$a_b + b_c = f(ab) - f(ac) - f(a)(b + c) - a(d(b) + d(c))$$
$$= f(ab) + f(ac) - f(a)(b + c) - ad(b + c)$$
$$= a_{b+c}$$

$\boxed{a_b + a_c = a_{b+c}}$ ⇒Proved (i)

(ii) Now

$$a^b = f(ab) - f(a)b - ad(b) \quad b^a = f(ba) - f(b)a - bd(a).$$

Then $a^b + b^a = f(ab) + f(ba) - f(a)b - f(b)a - ad(b) - bd(a)$

$$= f(a)b + ad(b) + f(b)a + bd(a) - f(a)b - ad(b) - f(b)a$$

$\Rightarrow \quad a^b + b^a = 0$

$\Rightarrow \quad \boxed{a}^b \boxed{= -b}^a$ Hence proved (ii)

■

Our aim is to show that if $a^b = 0 \; \forall\, a, b \in A$

$$\Rightarrow f(ab) = f(a)b - ad(b).$$

For this, we will use the following Lemmas

Lemma 4.2. *If $t \in V$ and t 6∈ Z and if $ut = tu$. Then $u \in V$*

Lemma 4.3. *If $a^2 = 0 \quad a \in A$ Then $a \in V$*

Lemma 4.4. *If $c,d \in V$ Then $b^a(cd - dc) = 0 \; \forall\, a, b \in A$*

5. Generalized Jordan Derivation becomes Generalized Derivation

Theorem 5.1. *If A is a prime ring of Ch 6= 2 Then any generalized Jordan Derivation becomes Generalized Derivation of A i.e.*

$$f(ab) = f(a)b + ad(b)$$

Proof. Let $u \in A$ satisfies $u^2 = 0$ By Lemma 4.3, $u \in V$ ■

Also if $x^2 = 0$, x is also in V. Then by Lemma 4.4

$$b^a(ux - xu) = 0 \quad \forall\, a\, b \in A$$

Post multiplying by u $b^a(ux - xu)u = 0$

$$\Rightarrow b^a(uxu - xu^2) = 0$$

$$\Rightarrow b^a uxu = 0 \quad (\because u^2 = 0)\; \forall\, a, b \in A. \quad (6)$$

Now for any $c,d \in A$

$$c^d(cd - dc) = 0$$

⇒ For any $r \in A$

$$((cd - dc)rc^d)^2 = 0$$

Let $u = (cd - dc)rc^d$ and $x = (ab - ba)sb^a$. Then from (6), we have $b^a(cd - dc)rc^d(ab - ba)sb^a(cd - dc)rc^d = 0\ \forall\ r,s \in A$.

Post multiplying by $ab - ba$

$\{b^a(cd - dc)rc^d(ab - ba)\}\ s\ \{b^a(cd - dc)rc^d(ab - ba)\}$

⇒ $b^a(cd - dc)rc^d(ab - ba) = 0$ (∵ A = Prime ring) $\forall\ r \in A$

⇒ either $b^a(cd - dc) = 0$ or $c^d(ab - ba) = 0\ \forall\ a,b,c,d \in A$ and A be Prime ring

Putting $d = b$. Then we get either $b^a(cb - bc) = 0$ or $c^b(ab - ba) = 0\ \forall\ a,b,c \in A$ (7)

By a result, we have

$$b^a(cb - bc) + c^b(ab - ba) = 0 \qquad (8)$$

From (7) and (8) it is clear that one or other term on LHS of (8) must be zero

⇒ $b^a(cb - bc) = 0 \quad a,b,c \in A.$

Now for any $a \in A$, $b^a \in T(b)$ and b 6∈ Z. Then

$T(b) = 0$ (By Lemma 2.3)

⇒ $b^a = 0 \quad \forall\ a \in A.$

On the other hand if $b \in Z$ and since $ba = ab\ \forall\ a \in A$. Then $b^a = 0\ \forall\ a \in A$
⇒ we can conclude that

$$b^a = 0\ \forall\ a\ b \in A.$$

Now

$b^a = -a^b = 0$

⇒ $\boxed{a}^b \boxed{= 0}$

⇒ $f(ab) = f(a)b + ad(b)$

⇒ f is Generalized Derivation Hence Proved.

Corollary 5.2. $f(ab) = f(a)b + ad(b)$ *This is Havala* [2] *Def. of Generalized derivation on P.1147.*

6. Generalized Jordan Derivation in Prime rings of $Ch = 2$ Now we re-define the generalized Jordan Derivation on any ring.

Definition 6.1. Let A be any ring. Then f is said to be Generalized
Jordan Derivation on A if it satisfies the following

(1) $f(a + b) = f(a) + f(b)$
(2) $f(ab) = f(b)a + bd(a)$
(3) $f(aba) = f(a)ba + ad(b)a + abd(a)$ where d = reverse derivation. Here (3) is derived from (1) & (2). Hence we have a theorem.

Theorem 6.2. ∴ *Let A be any Prime ring of Ch* = 2 *and if A is not commutative Integral Domain then any generalized Jordan Derivation is a Generalized Derivation.*

Proof. Combining Theorem 5.1 and 6.1 (Re-Definition), we get this result. ■

Conclusion

We showed that Generalized Jordan Derivatives of Prime Rings of Ch 6= 2 coincides with the Generalized Derivations. We also proved Herstein [3] Lemma 3.1 P.1106, Havala [2] Def. P.1147 as Corollaries of our results.

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