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Bilinear Transformation And Conformal Mapping

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ABSTRACT

We first look at bilinear transformation where we considered it as a combination of the transformations of translation, rotation, stretching and inversion. And we finally see that inversion mapping is $one - to - one$ and carries the class of straight lines and circles into itself. Which is property shared with translation, rotation and magnification. And we proceed to show that for a mapping $w = f(z)$ to be conformal in any domain, it is necessary and sufficient that $f(z)$ be analytic in that domain and that $f'(z) \neq 0$ for all z in the domain, and also shows that a mapping which preserves the magnitude of angles but not necessarily the sense is non-conformal rather Isogonal.

INTRODUCTION

Both from a geometric outlook and in terms of its plentiful application, one of the most appealing areas of this paper is the part devoted to the study of the univalent analytic functions. Functions of this type are also known as conformal mapping, for they are "angle preserving" in the sense of magnitude and direction.

The term conformal mapping is derived from geometric property of two dimensional mapping known as conformality.

This introduces us to bilinear transformations and the results concerning behavior of functions which are harmonic in the interior of a region, and differentiable on its boundary, under the change of variables determined by such a mapping.

BILINEAR TRANSFORMATION

Transformation of the form $W = \frac{az+b}{cz+d}$ ... ... (1.1)

Where a, b, c and d are in general, complex. And $ad - bc \neq 0$ is often called bilinear or fractional linear transformation. This transformation can be considered as combinations of the transformations of translation, rotation, stretching and inversion.

The transformation (1.1) has the property that circles in the $z - plane$ are mapped into circles in the $w - plane$, where by we includes circles of infinite radius which are straight lines.

1. Consider the line passing through the origin. The point $z = \rho e^{i\theta}$ is mapped to

$$f(z) = \frac{1}{z} = \frac{1}{\rho e^{i\theta}} = \frac{1}{\rho e^{i\theta}} \ (where\ \rho\ is\ real)$$

letting ρ vary, we see that the line making an angle $-\theta$ with the real axis, the point at ∞ goes to the origin and vice-versa.

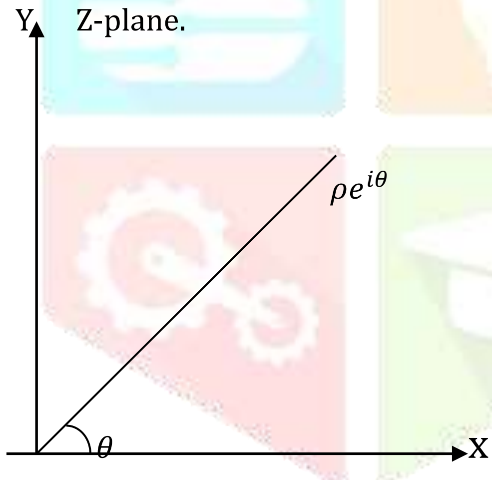

Figure .1

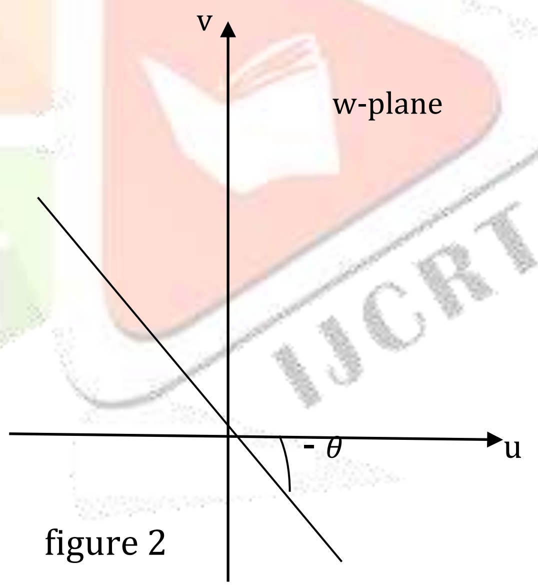

figure 2

2. Consider the line not passing through the origin in the $xy - plane$. The equation of such a line is given by $Ax + By = C$ ... ... (1.2)

With $c \neq 0$ since $f(z) = {}^{1}/_{z}$ we have on letting $w = f(z)$,

$$z = \frac{1}{w} = \frac{\bar{w}}{|w|^2} = \frac{u+iv}{u^2+v^2}; \quad where\ w = u + iv$$

And so $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$ ... ...(1.3)

Making the substitution into equation (1.2) we find that

$$A\left(\frac{u}{u^2+v^2}\right) + B\left(\frac{-v}{u^2+v^2}\right) = C$$

i.e. $u^2 + v^2 - \frac{Au}{C} + \frac{Bv}{C} = 0$ ... ... (1.4) completing the square yields

$$\left(u - \frac{A}{2C}\right)^2 + \left(v + \frac{B}{2C}\right)^2 = \frac{A^2 + B^2}{4C^2}$$

Which is a circle of radius $\frac{\sqrt{A^2 + B^2}}{2C}$ centered at $\left(\frac{A}{2C}, \frac{-B}{2C}\right)$

Hence the image points of the line (1.2) all lie on this circle (1.4) as shown below

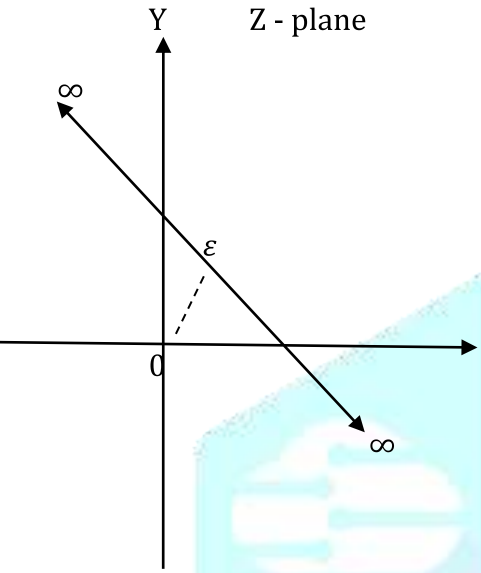

Figure 4

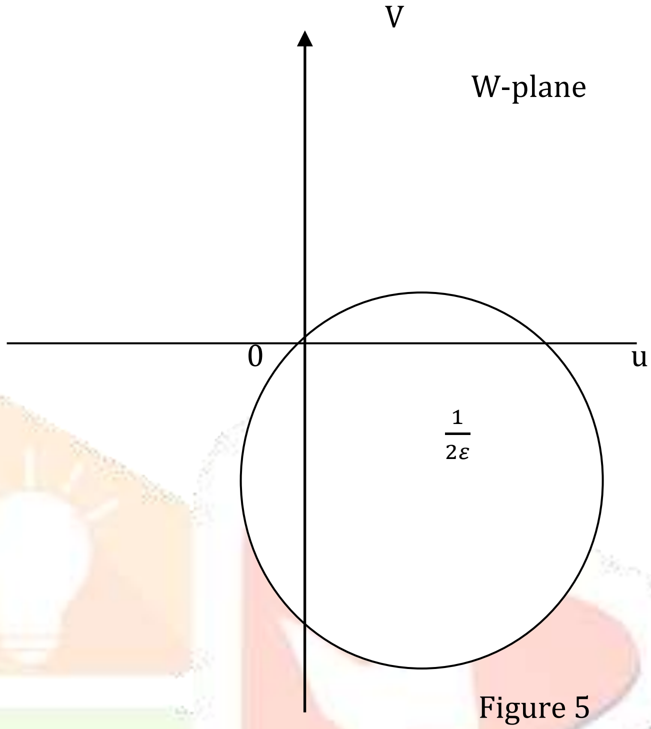

Figure 5

3. We consider a circle passing through the origin. Because the inverse of the map $w = \frac{1}{z}$ is itself an inversion, $z = \frac{1}{w}$, this case is just the reverse of the case (2). In other words, from case (2) the image of circle.

$$x^2 + y^2 - \frac{Ax}{C} + \frac{By}{C} = 0$$

Under the map $w = \frac{1}{z}$ is the line $Au + Bv = C$.

4. Now consider a circle not passing through the origin in the $xy - plane$, the equation of such a circle is of the form :

$x^2 + y^2 + Ax - By = C$ with $C \neq 0$. Using relation (1.3), this equation becomes

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{Au}{u^2 + v^2} - \frac{Bv}{u^2 + v^2} = C$$

Which simplifies to $1 + Au - Bv = C\,(u^2 + v^2)$. As $C \neq 0$, we therefore obtain

$$u^2 + v^2 - \frac{Au}{C} + \frac{Bv}{C} = \frac{1}{C}$$

For the equation satisfied by the image of the given circle, such an equation describes a circle in the $w-plane$ which does not pass through the origin, and as before, the image set is the whole circle.

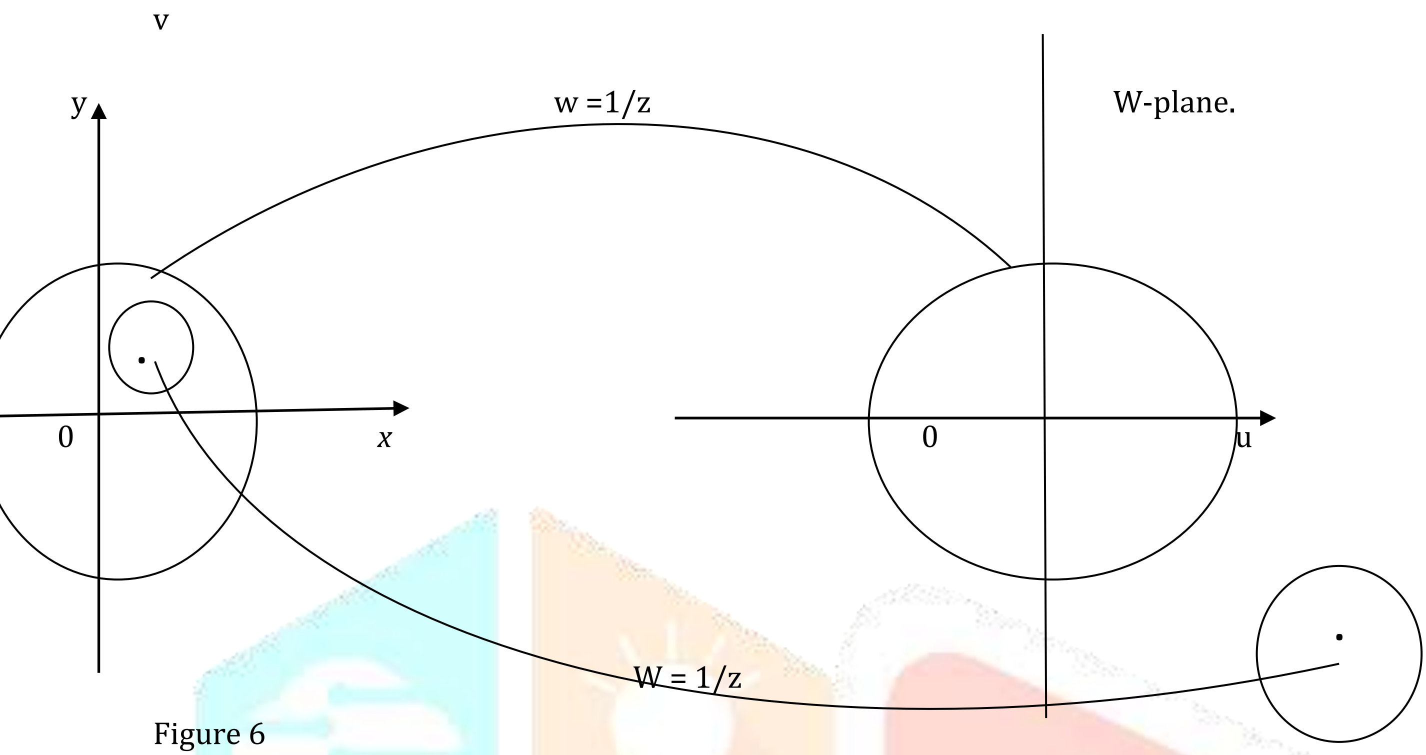

Figure 6

Therefore, we see that the inversion mapping is one-to-one and carries the class of straight lines and circles into itself, a property shared with translation, rotation and magnification

DEFINITION

A bilinear transformation (linear fractional or mӧbius transformation) is any function of the form $f(z) = \frac{az+b}{cz+d}$

With the restriction that $ad \neq bc$ (so that $f(z)$ is not a constant function). Notice that since $f(z) = \frac{ad-bc}{(cz+d)^2}$ does not vanish, the bilinear transformation $f(z)$ is conformal at every point except at pole $z = {}^{-d}/_{c}$. The bilinear transformation includes the elementary transformation as special cases. More important is the fact that any bilinear transformation can be decomposed into a succession of these elementary transformations. If $C = 0$, we have the linear transformation which was treated earlier. For $C \neq 0$, the decomposition can be seen by writing

$$\frac{az+b}{cz+d} = \frac{\frac{a(cz+d)}{c} - \frac{ad+b}{c}}{cz+d} = \frac{\frac{a}{c} + \frac{b-ad}{c}}{cz+d}$$

Which shows that the bilinear transformation can be expressed as a linear transformation (rotation + magnification + translation).

i.e. $f_1(z) = cz + d$ … … … … … … … … … … (1.5)

follows by an inversion

$$f_2(z) = \frac{1}{f_1} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (1.6)$$

And then another linear transformation

$$f(z) = \left(\frac{b - ad}{c}\right) f_2 + \frac{a}{c} \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (1.7)$$

As a result of this decomposition and of our previous deliberations, we have summarized some properties of bilinear transformation.

EXAMPLE

We consider a bilinear transformation which maps the upper half of the $z - plane$ into the unit circle in the $w - plane$ in such a way that $z = i$ is mapped into $w = 0$ while the point at infinity is mapped into $w = -1$.

We have $w = 0$ corresponding to $z = i$, and $w = -1$ corresponding to $z = \infty$.

Then from

$w = e^{i\theta_0}\left(\frac{z - z_0}{z - \bar{z}_0}\right)$ we have

$0 = e^{i\theta_0}\left(\frac{i - z_0}{i - \bar{z}_0}\right)$ so that $z_0 = i$ and $\bar{z}_0 = -i$, corresponding to $z = \infty$ we have $w = e^{i\theta_0} = -1$

Hence, the required transformation is

$$w = (-1)\left(\frac{z - i}{z + i}\right) = \left(\frac{i - z}{i + z}\right)$$

The situation is described graphically in figures below.

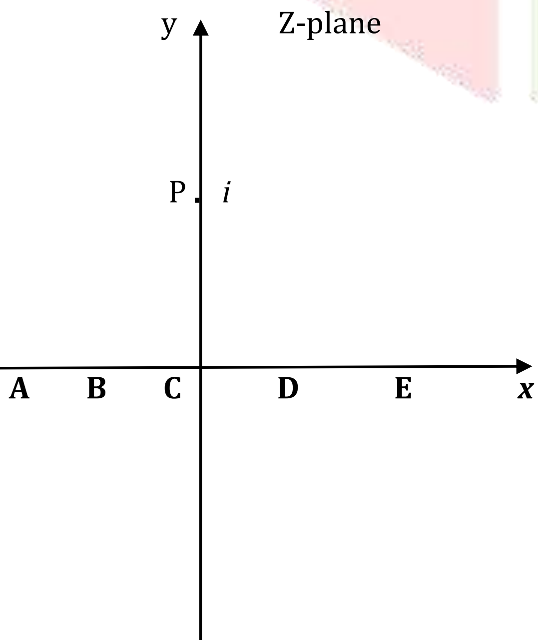

Figure 7

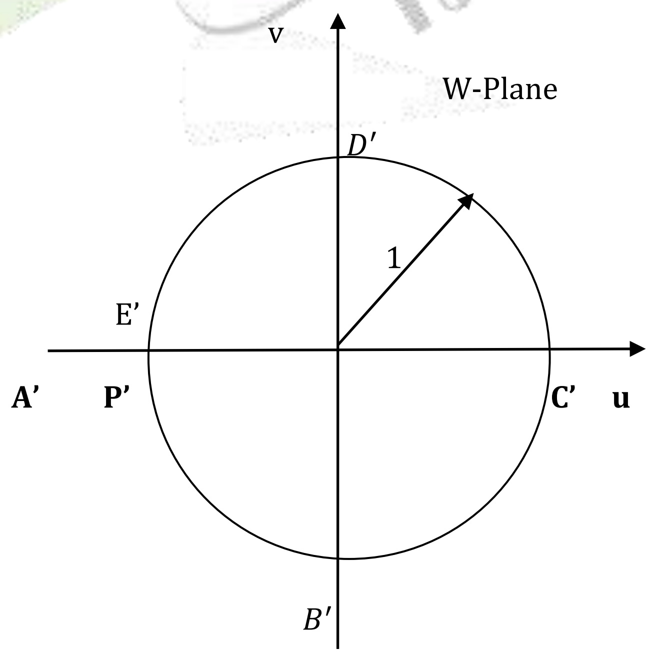

Figure 8

CONFORMAL MAPPING

The set of equations

$$U = u(x, y), \quad V = v(x, y)$$

Is defined, in general, as a transformation or mapping which establishes a correspondence between points in $w-plane$ and $z-plane$.

Suppose, under this transformation, point $z_0 = (x_0, y_0)$ of the $z-plane$ is mapped into $w_0 = (u_0, v_0)$ of the $w-plane$, while curves C_1 and C_2 (intersecting at z_0) are mapped, respectively, into curves C_1' and C_2' (intersecting at w_0). Then, if the transformation is such that the angle at z_0 between C_1 and C_2 is equal to the angle at w_0 between C_1' and C_2' both in magnitude and sense, then the transformation or mapping is said to be conformal at z_0. A mapping which preserves the magnitude of angles but not necessarily the sense is called ISOGONAL.

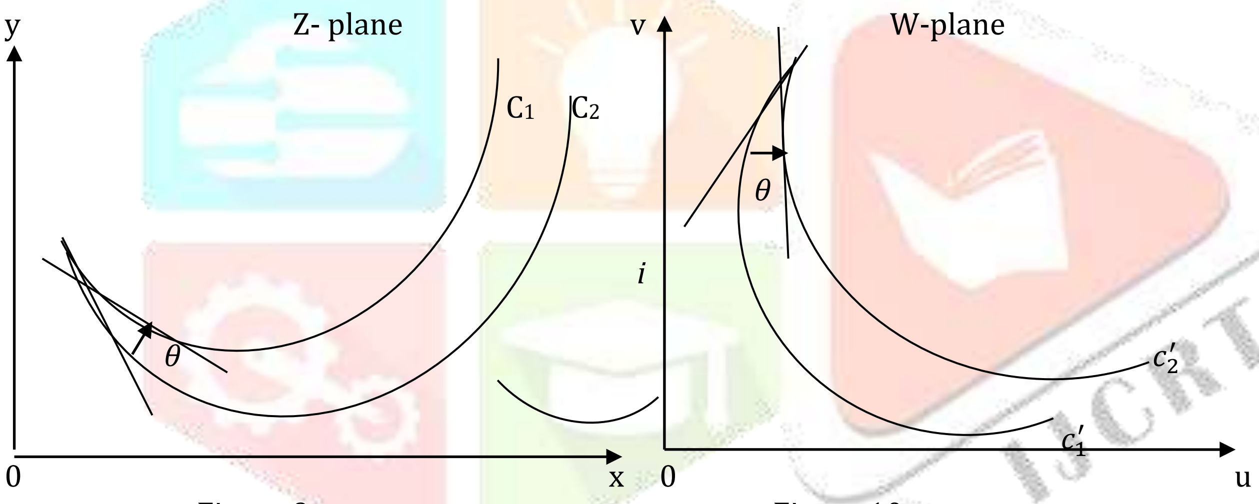

Figure 9 Figure 10

DEFINITION

A mapping from $z-plane$ onto $w-plane$ is said to be conformal if the angle between lines in the $z-plane$ are preserved both in magnitude and in sense of rotation when transformed onto the corresponding lines in the $w-plane$.

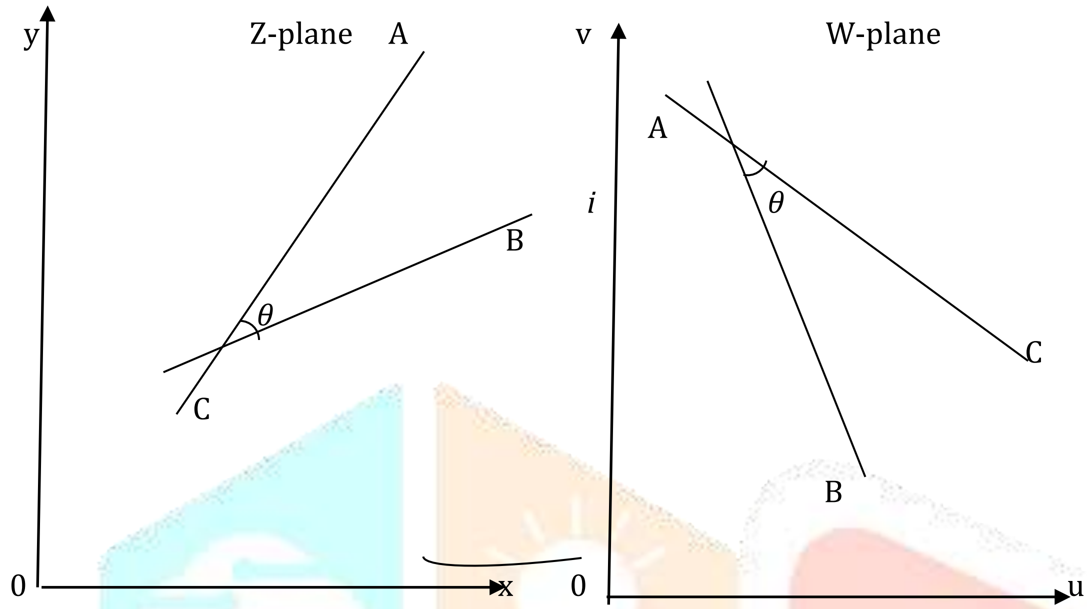

Figure 11 Figure 12

The angle between two intersecting curves in the $z-plane$ is defined by the angle $\alpha(0 < \alpha < \pi)$ between their two tangent at the point of intersection which is preserved.

The essential characteristics of a conformal mapping is that the angle is preserved both in magnitude and in the sense of rotation.

DEFINITION

The transformation $w = f(z)$, defined on a domain D, is referred to as a conformal mapping, when it is conformal at each point in D. That is, the mapping is conformal in D if it is analytic in D and $f'(z) \neq 0$.

EXAMPLE

We analyze that the mapping w = e^z throughout the entire $z-plane$, since,

$$\frac{\partial e^z}{\partial z} = e^z \neq 0 \text{ for each z.}$$

$$w = u + iv = e^z = e^{x+iy}$$

$$\Rightarrow \quad u + iv = e^x(\cos y + i \sin y)$$

$$\Rightarrow \quad u = e^x \cos y\,, \quad v = e^x \sin y \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.8)$$

Consider any two lines $x = C_1$ and $y = C_2$ in the $z-plane$, the first directed upwards while the second directed horizontally.

When $x = C_1$ (1.8) becomes $u = e^{C_1} \cos y\,,\ v = e^{C_1} \sin y$, Squaring both sides of the two equations and adding gives

$$u^2 + v^2 = e^{2C_1}$$

Which is a circle centered at the origin with radius e^{2C_1}

When $y = C_2$ (1.8) becomes $u = e^x \cos C_2$, $v = e^x \sin C_2$,

Dividing u by v gives: $\frac{u}{v} = \tan C_2$ or $v = (\tan C_2)u$, which is a straight line through the origin with slope $\tan C_2$, or a straight line through the origin making an angle of C_2 with the positive x – direction

Z - plane

W - plane

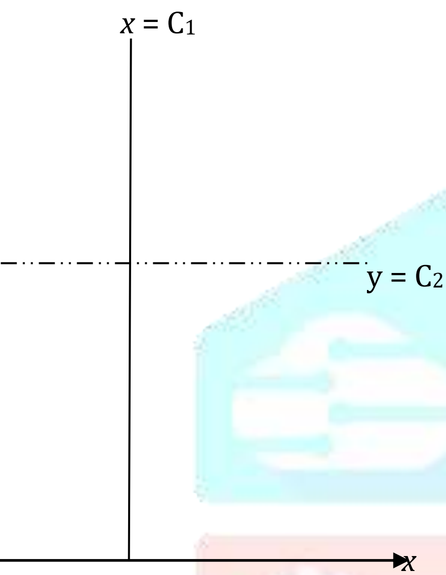

Figure 13

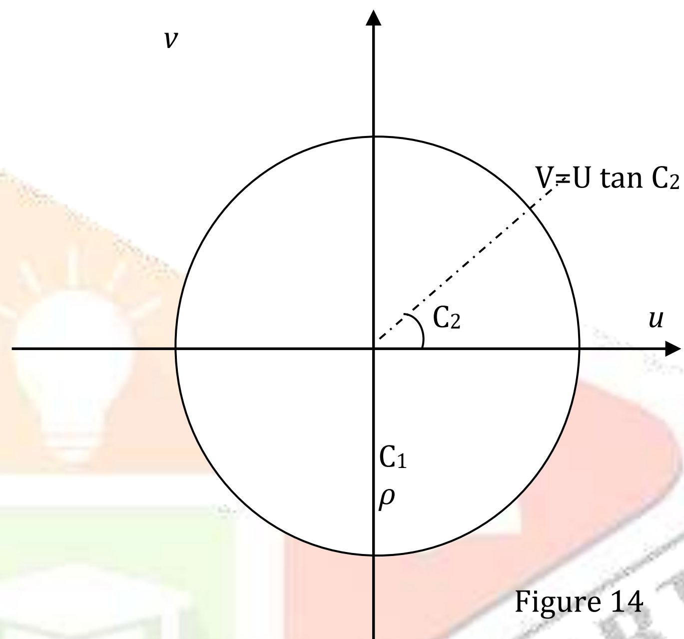

Figure 14

DEFINITION

A point at which $f'(z) = 0$ is called a critical point and at such a point, the transformation is not conformal. So if $w = f(z)$ is an analytic function, then except for points at which $f'(z) = 0$, the transformation function will preserve both the magnitude of its angle and sense of rotation. Suppose f is not a constant function and is analytic at a point Z_0 is called a critical point of the transformation $w = f(z)$.

EXAMPLE

The point $z = 0$ is a critical point of the transformation $w = 1 + z^2$

$w = f(z) = 1+z^2$

then $w = f(z) = 2z$, but $f(z) = 0$ since $z = 0$, by definition, if we write $z = re$

and $z^2 = r^2 e^{210} =$

Then, $w = 1 + \rho e$ where $= r^2$ and $\emptyset = 2\emptyset$

Hence, the point z = 0 is mapped onto the line$\emptyset = \alpha$ from the point w = 1 whose angle of inclination with the positive x – axis is 2α.

Moreover, the angle between any two lines drawn from the critical point z = 0is doubled by the transformation.

EXAMPLE

Consider a transformation w = f (z) where w = z is a reflection of the real axis Isogonal not conformal. The resulting transformation w = f (z) is Isogonal not conformal.

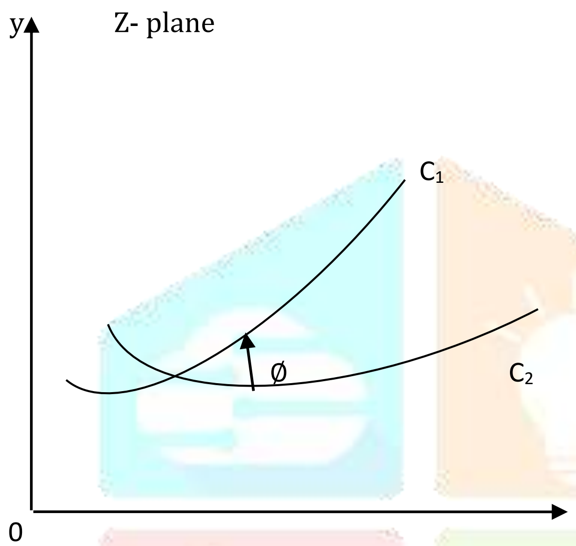

Figure 15

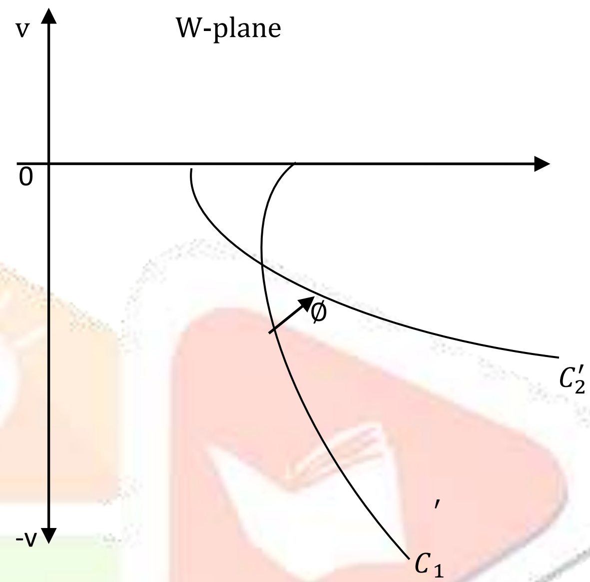

Figure 16

REIMANN MAPPING THEOREM

Any two plane, open simply –connected domains, whose boundaries consist of more than one point are conformally equivalent.

A (1-1) – mapping w= f (z) of the domain D in the z- plane onto the domain G of w- plane is said to be conformal, if this mapping at each point of D possesses or satisfies the conformality criterion.

CONFORMALITY CRITERION: for a mapping w = f (z) to be conformal in a domain D, it is necessary and sufficient that f(z) be analytic in D and that f (z) # 0 for all z in D.

We are now ready to turn our earlier definitions of an angle preserving transformation into a proper one. As announced, we shall restrict attention to deffomorphismss. A deffomorphisms f: D- C is said to angle preserving (also called Isogonal) at a point Z_0 of D under the condition that

Ø (A,B) = /Ø} (A), f (B), ″ ″ ″ ″ ″ ″ ″ ″ 3.9

Whenever A and B are sides of a curvilinear angle in D with vertex at z_0. It can b inferred from the soon to –come theorem that condition (3.9) allows for two eventualities.

If $J_f (z_0) > 0$, it reduces to

Ø (A,B)= Ø [f (A) ,f (B)] ″ ″ ″ ″ ″ ″ 3.10

For all relevant A and B;

If J_f $(z_0) < 0$, it becomes

$\emptyset$ (A,B) = $\emptyset$ [f (B), f(A)] " " " " " " 3.11

When (3.10) holds we say that f is conformal at Z_0 in the alternative situation (3.11) we pronounce f anti-conformal at z_0.

Thus, conformality demands that curvilinear angles be preserved not only in size, but also in sense. On the other hand, such angles are preserved in size but their orientation gets reversed.

THEOREM

Let D be a domain in the complex plane, and let F: D- C be a deffeomorphism. The following three statements concerning a point z_0 of D are equivalent.

1. F is differentiable at z_0:
2. F is Isogonal at z_0 and J_f (z_0) >0;
3. F is conformal at z_0.

There is an analogue of the above theorem treating the anti-conformal case.

The global version of the theorem marks "plane conformal mapping theory" as a special topic in the theory of analytic function.

THEOREM

Let D be a domain in the complex plane: A function F:D- C is a conformal mapping of D if and only if f is a univalent analytic function.

EXAMPLE

The function f (z) = (1-z) / (1+z) maps the disk D= D (0, 1) conformally onto the half plane. D'= {w: Re w>0}

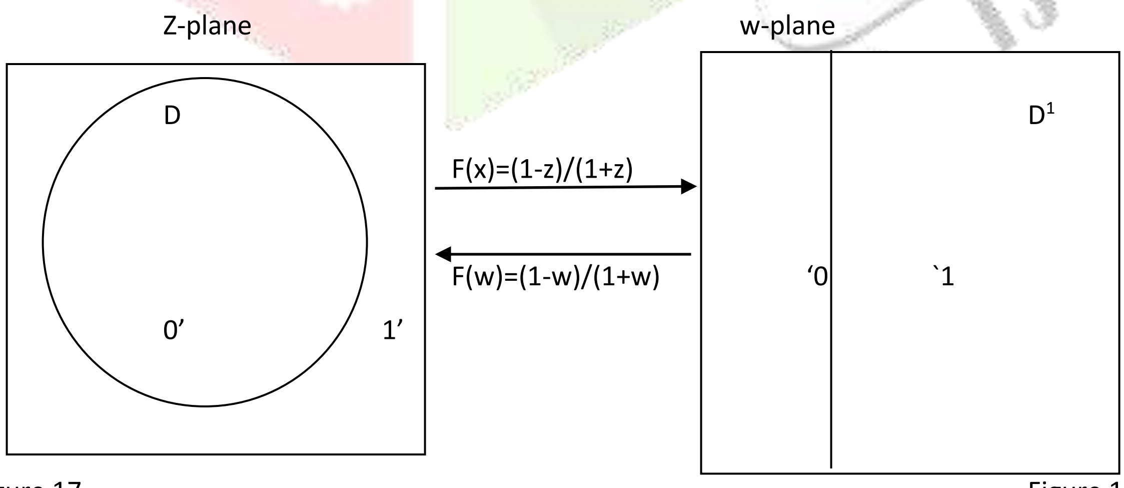

Figure 17 Figure 18

The function *f* is analytic in D and maps D in a univalent fashion onto D'

The above theorem certifies F as a conformal mapping of D onto D' Note that f^1 =f have, so f transforms D' conformally onto D as well.

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