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Uniform spaces

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Abstract :-

This paper deals with the concepts in the theory of uniform spaces. We also observe that the neighbourhood system of X for each U in the uniformity and consequently the family of all sets U [X] for U in v is the base for the neighbourhood. Since again we conclude that the uniformity in inherited by subsets of a uniform space by restriction.

Key-words: - *Uniform Structure, Subbase, Uniform Space, Interior, diagonal.*

Introduction :-

In the mathematical field of topology a uniform space in a set with a uniform structure. A uniform structure on a non-empty set X was first defined by A. Weil (1937) in terms of subsets of X × X. J.W. Tukey (1940) later provided as alternative description of a uniform structure using covers of X.

Basic concepts in the theory of uniform spaces:

Let *X be a* non-empty set. For arbitrary subsets U and V of $X\times X$, we write $V^{-1}=\{(y,x):(x,y)\in V\}$ and $U\,oV=\{(x,y):\exists Z\in X\}$ such that $(x,z)\in V$ and $(z,y)\in U\}$. It follows easily that $Uo(V\,o\,W)=(U\,o\,V)o\,W$ and $(U\,oV)^{-1}=V^{-1}o\,U^{-1}$. We shall write U^2 for $Uo\,U$. The diagonal of $X\times X$ which is denoted by $\Delta\,(x)$ or simply Δ defined as the set $\{(x,x) : x\in X\}$. For each subsets A of X the set $U[A]$ is defined to be $\{y: (x,y)\in U$ for some x in A $\}$. We write $U[x]$ tor $U[\{x\}]$ if x is a point in X. For each U and V and each A it is true that $(U\,o\,V)[A] = U[V[A]]$. Clearly $(U^{-1})^{-1} = U$, U *is* said to be symmetric if $U^{-1} = U$.

Definition:

A uniformity or uniform structure for a set X is a non-empty family $\mathcal{U}$ of subsets of $X\times X$ which satisfy the following conditions:

(i) Each member of $\mathcal{U}$ contains the diagonal Δ ;

(ii) if $U\in\mathcal{U}$, then $U^{-1}\in\mathcal{U}$;

(iii) If $U\in\mathcal{U}$, then $\exists V\in\mathcal{U}$ such that $V^2\subseteq U$;

(iv) If $U\in\mathcal{U}$ and $U\subseteq V\subseteq X\times X$, then $V\in\mathcal{U}$ and

(v) If U and V are members of $\mathcal{U}$, then $\bigcup\bigcap V \in \mathcal{U}$; Elements of $\mathcal{U}$ are said to be vicinities. A uniform space is a set together with a uniformity for it. Thus the pair $(X, \mathcal{U})$ is a uniform space.

Definition:

(i) A subfamily $\mathcal{B}$ for a uniformity $\mathcal{U}$ is a base for , iff each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.

(ii) *If* $\mathcal{B}$ is a base for $\mathcal{U}$; then $\mathcal{B}$ determines $\mathcal{U}$ entirely, for a subsets U of $X \times X$ belongs to $\mathcal{U}$ if U contains a member of $\mathcal{B}$.

Definition:

(i) A subfamily $\mathcal{B}$ is a subbase for $\mathcal{U}$ if the family of finite intersections

(ii) of members of $\mathcal{B}$ is a base for $\mathcal{U}$.

(iii) We now state the following theorem, the proof of which is simple.

Theorem :

A non-empty family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity for X if and only if

(i) Each member of $\mathcal{B}$ contains the diagonal Δ;

(ii) If $U \in \mathcal{B}$, then $\exists V \in \mathcal{B}$ such that $V \subseteq U^{-1}$;

(iii) If $U \in \mathcal{B}$, then $\exists V \in \mathcal{B}$ such that $V^2 \subseteq U$;

(iv) If $U, V \in \mathcal{B}$ then $\exists W \in \mathcal{B}$ *such that* $W \subseteq \bigcup\bigcap V$.

Theorem:

A family $\mathscr{b}$ of subsets of $X \times X$ is a subbase for some uniformity for X if

(i) each member of $\mathscr{b}$ contains the diagonal Δ;

(ii) for each $U \in \mathscr{b}$, $\exists V \in \mathscr{b}$ such that $V^2 \subseteq U^{-1}$.

(iii) for each $U \in \mathscr{b}$, $\exists V \in \mathscr{b}$ such that $V^2 \subseteq U$.

In particular, the union of any collection of uniformities for X is the subbase for a uniformity for X.

Proof:

We have to show that the family $\mathcal{B}$ of finite intersections of member of $\mathscr{b}$ satisfies the condition of theorem (5.1).

If $U_1, U_2, \ldots\ldots\ldots\ldots U_n$ *and* $V_1, V_2, \ldots\ldots\ldots\ldots\ldots V_n$ are subsets of $X \times X$ all belonging to $\mathscr{b}$ and if $U = \bigcap_{i=1}^{n} U_i$ and $V = \bigcap_{i=1}^{n} V_i$ then $V \subseteq U^{-1}\left(or\, V^2 \subseteq U\right)$ whenever $V_i \subseteq U_i^{-1}$(respectively, $V^2{}_i \subseteq U_i$) for each i. From this observation the proof of this theorem follows.

Definition:

If $(X, \mathcal{U})$ is a uniform space the topology J of the uniformity $\mathcal{U}$, or the uniform topology is the family of all subsets T of X such that for each $X \in T$ there is $U \in \mathcal{U}$ such that $U[x] \subseteq T$.

To verify that J is a topology is simple. In fact the union of members of *J* is surely a member of *J*. If T and *S are* members of *J* and $x \in T \cap S$, there are *U* and *V* in $\mathcal{U}$ such that $U[x] \subseteq T$ *and* $V[X] \subseteq S$, and hence $U \cap V[X] \subseteq T \cap S$ consequently $T \cap S \in J$ and *J* is a topology.

Theorem:

The interior of a subset *A* of *X* relative to the uniform topology is the set of all points *X* such that $U[X] \subseteq A$ for some *U* in $\mathcal{U}$.

Proof:

To prove the theorem it is sufficient to prove that the set $B = \{X : U[X] \subseteq A$ for some U in $\}$ is open relative to the uniform topology, for *B* surely contains every open subset of *A* and, if *B* is open, then $\exists U \in \mathcal{U}$ such that $U[X] \subseteq A$ and again $\exists V \in \mathcal{U}$ suchthat $V^2 \subseteq U$. If $y \in V[X]$ then $V[y] \subseteq V^2[X] \subseteq U[X] \subseteq A$ and $y \in B$. hence $V[X] \subseteq B$ and B is open.

This completes the proof.

Remark:

It follows immediately that $U[X]$_ is a neighbourhood system of *x* for each *U* in the uniformity $\mathcal{U}$, and consequently the family of all sets $U[X]$ for *U* in $\mathcal{U}$ is a base for the neighbourhood system of *x* (the family is actually identical with the neighbourhood system). The following theorem is then clear.

Theorem:

If $\mathcal{B}$ is a base (or subbase) for the uniformity $\mathcal{U}$, then for each *x* the family of sets $U[X]$ for *U* in $\mathcal{B}$ is a base (subbase respectively) for the neighbourhood system of *x*.

Remark (2) :

A uniformity is inherited by subsets of a uniform space by restriction.

If *X* is a uniform space for a uniformity $\mathcal{U}$, and *Y* is a subset of *X*, then *Y* is a uniform space (called subspace) under the induced (relative) uniformity

$Y_u = \{YxY \cup U : U \in \mathcal{U}$ for Y.

If $\mathcal{B}$ is a base for , then $Y_B = \{YxY \cup U : U \in B\}$ is a base for*Y*. It can be verified that the topology of the relative uniformity ϑ is the relativized topology for.

Conclusion: Hence , the interior of a subset A of X relative to the uniform topology is the set of all points X such that $U[X] \subseteq A$ for some U in $\mathcal{U}$ and also a uniformity is inherited by subsets of uniform space by restriction.

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