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On the Number of Minimal Subgroups and Theta Pairs in a Finite Groups

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Abstract: Let M be a minimal subgroup of a finite group G. The pair (A, B) of subgroups of G is called a θ-pair of M if the following condition hold: (a) $B \lhd G, B < A$, (b) $< M, A >= G$ and $B \leq M$ and (c) $\frac{A}{B}$ has no proper normal subgroup of $\frac{G}{B}$. In this paper, we investigate the structure of a finite group G according to the number of its minimal subgroups and θ-pairs.

Keywords: finite group, theta pair, minimal subgroup.

I. INTRODUCTION

In this paper all groups considered are assumed to be finite groups, For convenience we denote $M < G.$ to indicate that M is minimal subgroup of a group G. Also, M_G denotes the core of M in G and $\varphi(G)$ is the frattini subgroup of the group G.

In [7], Mukherjee and Bhattacharya introduced the concept of θ-pairs associated to minimal subgroups of a group, and used this concept to investigate the structure of some groups. Then Beidlemen and Smith [3], generalized the concept to the universe of infinite groups. The investigtion on θ-pairs are continued in [2,8] and [11-15]. Let us recall the definition of θ-pairs which is introduced by Mukherjee and Bhattacharya.

II.PRELIMINARIES

Definition 1.1. Given a minimal subgroup *M* of a group *G*, a θ-pair of *M* is any pair (*A*, *B*) of subgroups satisfying the following conditions:

(a) $B \lhd G, B < A$

(b) $< M, A >= G$ *and* $B \leq M$

(c) (c) $\frac{A}{B}$ *has no proper normal subgroup of* $\frac{G}{B}$.

In addition, if A ⊴ G, then (A,B) is called a normal θ-pair. A θ-pairs (A,B) is said to be minimal if there is no θ-pair (C,D) such that A <C. The nonempty set of all θ-pairs of M in G is denoted by θ(M) and θ(G) = $\bigcup_{M<G} \theta(M)$.

The aim of this paper is to investigate the problem of existence of finite groups with a given number of theta pairs. From the definition of θ(G), one can see that the number of θ−pairs in a finite group is related to the number of minimal subgroups of the group under consideration. So it is natural to investigate the same problem for minimal subgroups.

It is well known that if a finite group G has exactly one minimal subgroup, then $|G|$ is divisible by exactly one prime number and G is cyclic. In this connection one might ask about the structure of G, if G has exactly two or three minimal subgroups.

A group G has exactly two minimal subgroups then $|G|$ is indeed divisible by two primes and G is cyclic, and if G has exactly three minimal subgroups then neither G needs to be cyclic nor it is required for $|G|$ to be divisible by three primes. In fact, in this case G is a 2-group or a cyclic group with exactly three prime factors, see for details [6].

Definition 1.2. Let n be a natural number. By $\mathcal{L}$(n) we denote the set of all non-isomorphic finite groups with exactly n minimal subgroups. We define a binary relation $\precsim$ on $\mathcal{L}$(n) as follows:

$$H \precsim G \Leftrightarrow \exists\, N \trianglelefteq G \quad s.t\ \frac{G}{N} \cong H.$$

III. Main Result

Corollary 2.1. Let M be a minimal subgroup of the group tt. Then, for all $g \in G, |\theta(M)| = |\theta(M^g)|$

Proof. The map $\tau: \theta(M) \to \theta(M^g)$ that sends (C,D) to (C^g,D) is well-defined. Now, it is easy to see that the map τ is a one-to-one correspondence.

Let G be a finite group and M be a minimal normal subgroup of G. Then (G,M) is a θ- pair of M in G. So $\theta(M) \neq \emptyset$ .In what follows, we investigate the structure of finite groups with exactly 1 and 2 $\theta -$ pair

Lemma 2.2. A group G has exactly one θ −pair if and only if G is a cyclic group of prime power order.

Proof. Suppose G has exactly one θ −pair. Then $\frac{G}{\varphi(G)}$ is a simple group and $\theta(G) = \{(G, \varphi(G))\}$. Suppose m(G) > 1. Then $\varphi(G)$ is not minimal in G and for any minimal subgroup M of G, (M, $\varphi(G)$) is a $\theta -$ pair of L, in which L is a minimal subgroup of G distinct from M, a contradiction. This shows that $m(G) = 1$ and so G is a cyclic group of prime power order.

Lemma 2.3. If there exists a minimal subgroup M of G such that $\theta(M) = \theta(G)$,then G has exactly one θ- pair.

Proof. It is obvious that G has exactly one θ-minimal and so $\frac{G}{\varphi(G)}$ is a simple group. If m(G) > 1 then (M, $\varphi(G)$) ∈ $\theta(L)$ and (L, $\varphi(G)$) ∈ $\theta(M)$, for two distinct minimal subgroups M and L of G, which is a contradiction. Therefore, $m(G) = 1$ and by lemma 2.2 , G has exactly one θ-pair, proving the lemma.

Lemma 2.4. There is no $n\theta$ −pair cyclic group of order $p_1^{r_1}.p_2^{r_2}$ .....$p_n^{r_n}, p_1 < p_2 < \cdots < p_n$, $in\ which\ \ n > 1$.

Proof. Suppose $\{M_1, M_2, \dots . M_n\}$ is the set of all minimal subgroups of G. Then

(G, M_i), $1 \leq i \leq n,$ are $n$minimal θ-pairs for G and so G has at least n θ -pair. Assume that M is a minimal subgroup of index p_1, A is a minimal subgroup of M of index p_2 and L is a minimal subgroup of G of p_2. Then (M,A) ∈ θ(L) a contradiction.

We now are ready to state one of our main results. We have:

Theorem 2.5. There is no finite group with exactly two θ-pairs.

Proof. Let G has exactly two θ-pairs. By Lemma 2.3, there is no minimal subgroup M of G such that $\theta(M) = \theta(G)$ and so G has exactly two minimal θ-pairs. Thus $|\{X_G | X <. G\}| = 2$. Suppose that (C,L_G) and (G,M_G) are two distinct minimal θ-pairs of G associated to minimal subgroups L and M, respectively. We claim that G has exactly two minimal subgroups. To do this, we assume that T is a minimal subgroup of G different from M and L. if C≠ G , then $\varphi(G) = L(G)\ and\ \ (L.\varphi(G)) \in \theta(T)$ .which is a contradiction. We now assume that C = G, then $\frac{G}{M_G}$ and $\frac{G}{L_G}$ is a simple groups, Therefore $T_G = L_G$ or $T_G = M_G$ .Suppose $T_G = L_G$ Then $(L, L_G) \in \theta(T)$, a contradiction .Also ,if $T_G = M_G$ then $(M, M_G) \in \theta(T)$ and so $M_G = L_G$ This implies that $\frac{G}{\varphi(G)}$ is a simple group. which is a contradiction. Therefore, G has exactly two minimal subgroups and so $|G|$ is indeed divisible by two primes. Now by Lemma 2.4, the proof is complete.

Lemma 2.6. let G be a finite group such that all of minimal θ −pair pairs of G are normal and $\{M_G|M <. G\} = \{L_{1G}, \ldots . L_{rG}\}$.Then $\theta_{max}(G) = \theta_{max}(L_1) \cup \ldots \cup \theta_{max}(L_r)$.

Proof. Suppose (C,D) is an arbitrary minimal θ −pair of G. Then D= L_{iG} for some $1 \leq i \leq r$, If $C \subseteq L_i$ $then\ C \subseteq D$, a contradiction. Thus (C,D) $\in \theta(L_i)$ Now we assume that (E,F) is a minimal θ-Pair of $\theta(L_i)$ such that (C,D) $\leq (E,F)$. $Therefore$ $C \leq E, D = F$, $\frac{C}{D} \leq \frac{E}{D}$ and $\frac{C}{D} \trianglelefteq \frac{G}{D}$. This shows that (C,D) is a minimal θ-pair of $\theta(L_i)$ an the proof is complete.

In the following theorem, we prove that there is no also finite groups with exactly three θ-pair.

Theorem 2.7. There is no finite group with exactly three θ-pairs.

Proof. Let G be a 3θ-pair group. There is no minimal subgroup M of G such that $\theta(M) = \theta(G)$). Our main proof will consider a numbers of cases.

Case 1. There are two minimal subgroups M and L of G such that $|\theta(M)| = 2\ and\ |\theta(L)| = 1$

Assume that $(B, M_G), (C, D) \in \theta(M)$ and $(A, L_G) \in \theta(L)$. we can see that C ⊴ D and G ≠ C .We claim that G has at least three minimal subgroups. By lemma 2.2., G has at least two minimal subgroups. Assume that G has exactly two minimal subgroups, say M and L. Thus, by the mentioned theorem of khazal, G is cyclic and so $(A, L_G) = (G, L), (B, M_G) = (G, M)$. Since $\frac{G}{L}$ is a simple group, we have $(M, \varphi(G)) \in \theta(L)$ a contradiction. Therefore G has at least three minimal subgroups. We now see that $M_G \neq L_G$. Thus, for any minimal subgroup X of G , $X_G = L_G$ or $X_G \leq M_G$, Suppose A= G. If L is non-normal and g $\in G - N_G(L)$ then $(L^g, L_G) \in \theta(L)$ which is impossible. So L⊴ G and we can see that $(M_{G,} L \cap M_G) \in \theta(L)$, a contradiction. Thus A$\neq G$ $and\ so\ A \leq M_G. Also\ C \leq L_G\ and\ hence\ C \leq L_G \leq A \leq M_G$, which is a contradiction.

Case 2. G is 3θ-minimal and there are minimal subgroups M, L and K of G such that

$(A, L_G) \in \theta(L)$, $(B, K_G) \in \theta(K)$ and $(C, M_G) \in \theta(M)$ by lemma 2.6 and case 1,

$|\{M_G|M <. G\}| = 3$. We claim that one of the subgroups A,B and C is equal to G and the other two are proper. To do this, suppose A = C = G. Then M , L ⊲ G and $(L, M \cap L) \in \theta(M)$ which is impossible. Therefore, we can assume that A ≠G , B$\neq G$ and $\left|\theta(\frac{G}{A})\right| = \left|\theta(\frac{G}{B})\right| = 1$. Suppose $\frac{R}{A}$ and $\frac{S}{B}$ are the unique minimal subgroup of $\frac{G}{A}$ and $\frac{G}{B}$ respectively. Thus $(\frac{G}{A}, \frac{R}{A}) \in \theta(\frac{G}{A})$ And $(\frac{G}{B}, \frac{S}{B}) \in \theta(\frac{G}{B})$. This shows that (G,R) are (G,S) are θ-pairs we can assume that M ⊲ G and A,B $\leq M$, Now $\left(\frac{A}{L_G}, \frac{L_G}{L_G}\right), \left(\frac{G}{L_G}, \frac{M}{L_G}\right) \in \theta\left(\frac{G}{L_G}\right)$ and $\left|\theta_{max}(\frac{G}{L_G})\right| \leq 3$ Therefore $\left|\theta_{max}(\frac{G}{L_G})\right| = 3$ and there exists another θ-pair $\left(\frac{R_1}{L_G}, \frac{U_1}{L_G}\right) \in \theta\left(\frac{G}{L_G}\right)$.It is easy to see that $L_G \subseteq K_G$. Using similar argument as in above , $K_G \subseteq L_G$ $and\ so\ L_G = K_G$. . which is a contradiction.

Theorem 2.8. There exists a group with exactly n θ-pair for n ≠2,3.

Proof. For n =1, a cyclic group of prime power order has exactly one θ-pair Suppose $n \geq 4$ and $G = Z_{p^n q}$. Then G has exactly two minimal subgroups M and N of orders p^n and $p^{n-1}q$, respectively. Suppose A_i and B_i , $0 \leq i \leq n$, subgroups of G order p^i and $p^i q$. now it is easy to see that $\theta(M) = \{(B_i, A_i)|0 \leq i \leq n$ and $\theta(N) = \{(A_n, A_{n-1}), (B_n, B_{n-1})\}$.Therefore G has exactly $n + 3$, proving the result.

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