IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Persistence in a Ratio Dependent Food Chain Model with Facultative Mutualism

Vimlesh[1], Rachana Pathak[2] and Manju Agarwal[3]

[1]Department of Mathematics & Statistics, Shri Ramswaroop Memorial University, Lucknow-Deva Road, Barabanki-225003, India

[2]Department of Applied Science (Mathematics), Faculty of Engineering & Technology, University of Lucknow, New Campus, Lucknow-226031, India

[3]Department of Mathematics & Astronomy, Lucknow University, Lucknow-226007, India

Abstract

This paper is concerned with a three species food chain model whose population interacts with a mutualist. Holling type II functional response is considered to model the interactions between the prey and middle predator and the interaction between the middle predator and top predator is assumed to be governed by ratio dependent functional response. The mutualism is facultative in which mutualism occurs with the prey population, modelled by system of four autonomous ordinary differential equations. Conditions of the boundedness of system are established. Existence of equilibrium point and stability analysis of model is carried out by using usual theory of ordinary differential equations. It is found that if the maximal growth rate of the top predator is less than the death rate then top predator face extinction. Also we obtained the conditions that influence the persistence of system. Furthermore global stability of the system is shown graphically. It has been observed from numerical simulations that in the presence of mutualist population, prey increases and hence population of predator also increases, but in the absence of mutualism, population of prey decreases and hence predators population also decreases.

Keywords:- Food chain, Facultative mutualism, Ratio-dependent, Boundedness, Extinction, Stability, Persistence.

1. Introduction

All species on earth are closely related to other species. In a simple view the interaction between the pair of species can be classified into two categories: predation (one gains and other suffers), competition and mutualism. In recent years, the concern for mutualism is growing since most of the world biomass is dependent on mutualism (Bashary and Branstein 2004). For example microbial species influence on the abundances and ecological function of related species (Holland et al., 2010). Many bacterial species co-exist in a syntrophic association (obligate mutulism); that is, one species cannot survive without the mutualist species. Whereas a facultative mutualists, benefitting from the presence of each other, may also survive in absence of each other. Mutualism is part of many significant systems and processes, such as mycorrhizol association, nitrogen fixation, gut faunas and floras, endosysmbiotic photosynthesis, pollination, endozoic algae etc ..

Most models of mutualism are two dimensional (Dean (1983), Freedman (1987), Freedman et al. (1983), Boucher (1995)). There has been a fair amount of work on three dimension models, were the mutualism occurs between prey and predators perhaps both, or competitors were analysed by Rai et al. 1983 and expanded by Kumar and Freedman 1989. There is very little work on four dimensional mutualistic models. Rai and Freedman (1995), Freedman et al. (2001) considered mathematical models arising due to interaction of two competitors each with a mutualist have been modelled. Kumar and Freedman (1989 and 2001) has been analysed a food chain model in which facultative and obligative mutualism interactions occurs with the prey and first predator population. Compared to previous two, three and four dimensional models of mutualism, our model represent a more structure of facultative mutualistic systems, the mechanism by which one species benefit another, and effect of this beneficial interaction on first and top predators. Derivation of the model by basic principle of biology and the results obtained provide an important link between mathematical models and the eventual understanding of the dynamics of real world system. Here we address the question for the case in which existing community consist of two predators in which first predator capture only prey, top predator prey only first predator and prey is in a beneficial interaction with a mutualist. For example Pseudomyrmex- ant and Acacia tree mutualism. The ants defend these small trees against herbivorous, insect and vertebrates. The ants also chew away and sting any encroaching plants, clearing an area that may be up to 4yd(4m) in radius. In return, the plant gives ants the food they like. The plants also produce thorn that the ants hollow out for nests. These ants are eaten by Cyclopes did actyuls (silky anteaters) –harpy eagle eats anteater.

The rest of the paper is organized as follows. In section 2, we present a brief sketch of the construction of the model. The boundedness of our model is studied in section 3. In section 4 and 5, we have determined the boundary equilibrium point and their stabilities, it is seen that the mutualist-free boundary equilibrium point possesses nonempty stable and unstable manifold whenever the interior equilibrium point exists. Conditions which influence the persistence of system is studied in section 6. Computer simulations are performed to illustrate the feasibility of our analytical findings in section 7. At last general discussion of the paper is presented in section 8.

2. Mathematical Model

In this section we describe a model of interactions between a mutualist population and population of food chain. The mathematical formulation of facultative relationships between the mutualist and the prey of the food chain is described. Mathematically this model can be represented by the following system of autonomous differential equations:

$$\begin{aligned}
\dot{u}(t) &= u\left(1-\frac{u}{L+lx}\right),\\
\dot{x}(t) &= \alpha x\left(1-\frac{x}{K(u)}\right)-\frac{c_1xy}{a_1+x+mu},\\
\dot{y}(t) &= \frac{p_1xy}{a_1+x+mu}-q_1y-\frac{c_2yz}{y+z},\\
\dot{z}(t) &= -q_2z+\frac{p_2yz}{y+z}.
\end{aligned} \tag{2.1}$$

Define $K(u)=k_0+ku$, $u(0)>0$, $x(0)>0$, $y(0)>0$ and $z(0)>0$.

Here we have following assumption:

1. The mutualist growth is of logistic type.
2. The prey growth in the absence of predators is logistic and dependent of the mutualist.
3. Predation by the middle predator in the absence of mutualism is Holling type predation.
4. A ratio- dependent functional response is considered for top predator population.

Where t represent the time, u is the density of mutualist population at any time t, x(t), y(t), and z(t) denote the prey, predator and super predator densities respectively. The model parameters are assuming only positive values. For $i=1,2$ q_i, p_i and c_i are natural death rate of predators, conversion rate for predation of predators and capturing rates of predators respectively. L and k are the carrying capacity of mutualist and prey population. α being the intrinsic growth rate of prey and l, m are mutualism constants. The above assumptions are ecologically reasonable and exemplified.

3. Boundedness

In theoretical biology, boundedness of a system implies that the system is biologically well behaved. The following theorem ensures the boundedness of the system (2.1).

Theorem (3.1):- The set

$$B=\{(u,x,y,z):0\le u\le L+l\tilde{k},0\le x\le \tilde{k},0\le \frac{p_1}{c_1}x+y+\frac{c_2}{p_2}z\le \frac{M}{\ell}+\varepsilon,\text{for any }\varepsilon>0\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$\tilde{k}=\max(K(u)), \qquad M=\frac{2p_1}{c_1}\alpha\tilde{k}.$$

Proof: - From the first equation of system (2.1), we have

$$\dot{u}(t)\le u\left(1-\frac{u}{L+l\tilde{k}}\right),$$

Hence, $\lim\limits_{t\to\infty}\sup u(t)\le L+l\tilde{k}$.

Where $\tilde{k}=\max(K(u))$.

From the second equation of system (2.1), we get

$$\lim_{t\to\infty}\sup x(t)\le \tilde{k}.$$

Let us define,

$$W = \frac{p_1}{c_1}x + y + \frac{c_2}{p_2}z,$$

Then,

$$\frac{dW}{dt} = \frac{p_1}{c_1}\alpha x\left(1 - \frac{x}{k_0 + ku}\right) - q_1 y - \frac{q_2}{p_2}c_2 z$$

$$\le \frac{p_1}{c_1}\alpha x - \min\{q_1, q_2\}\left(y + \frac{c_2}{p_2}z\right),$$

$$\le \frac{2p_1}{c_1}\alpha\tilde{k} - \ell W, \text{where } \ell = \min\{\alpha, q_1, q_2\}$$

Therefore,

$$\frac{dW}{dt} + \ell W = M.$$

Applying a theorem on differential inequalities {Birkhoff and Rota (1982)}, we obtain

$$0 \le W(x, y, z) \le \frac{M}{\ell} + \frac{W(x(0), y(0), z(0))}{e^{\ell t}},$$

and for $t \to \infty$,

$$0 \le W \le \frac{M}{\ell}.$$

Thus, all solutions of (2.1) enter into the region:

$$B = \left\{(u, x, y, z) : 0 \le u \le L + l\tilde{k}, 0 \le x \le \tilde{k}, 0 \le W \le \frac{M}{\ell} + \varepsilon, \text{for any } \varepsilon > 0\right\} \tag{3.1}$$

Hence complete the theorem.

4. Equilibria: its existence and stability

The equilibria for our model are determined by solving following equation given below:

$$u\left(1 - \frac{u}{L + lx}\right) = 0,\ \alpha x\left(1 - \frac{x}{K(u)}\right) - \frac{c_1 xy}{a_1 + x + mu} = 0,\ \frac{p_1 xy}{a_1 + x + mu} - q_1 y - \frac{c_2 yz}{y + z} = 0, \tag{4.1}$$

$$-q_2 z + \frac{p_2 yz}{y + z} = 0.$$

There are at most four non negative equilibrium points for our model

1. In the absence of mutualist population and top predator population, the middle predator can survive on its prey. Therefore equilibrium point $\overline{E}(0, \overline{x}, \overline{y}, 0)$ is given by

$$\overline{x} = \frac{q_1 a_1}{p_1 - q_1},\ \overline{y} = \frac{\alpha(a_1 + \overline{x})(k_0 - \overline{x})}{k_0 c_1}.$$

For existence, $p_1 > q_1$.

Due to extinction of middle predator there is no equilibrium point in the $x - z$ plane.

Moreover neither y nor z can survive in the absence of prey species x, hence there is no equilibrium point in the $y - z$ plane.

2. The positive equilibrium point $\hat{E}(\hat{u},\hat{x},\hat{y},0)$ exists in the interior of the first octant if $p_1 > q_1(1+ml)$ and given by

$$\hat{x} = \frac{q_1(a_1+mL)}{p_1 - q_1(1+ml)}, \qquad \hat{u} = L + \frac{lq_1(a_1+mL)}{p_1 - q_1(1+ml)},$$

$$\hat{y} = \frac{\alpha}{c_1}\left(\frac{(k_0+kL)(p_1-q_1(1+ml)+lkq_1(a_1+ml)-(a_1+mL)q_1}{(k_0+kL)(p_1-q_1(1+ml))+klq_1(a_1+mL))}\right)\left(\frac{(a_1+mL)}{p_1-q_1(1+ml)}\right).$$

It is positive if $(k_0+kL)(p_1-q_1(1+ml) > (a_1+mL)q_1(1-kl)$. (4.2)

3. In the absence of mutualist population all the species can survive. Hence equilibrium point $\breve{E}(0,\breve{x},\breve{y},\breve{z})$ exist and given by

$$\breve{z} = \left(\frac{p_2-q_2}{q_2}\right)\breve{y}$$

$$\breve{x} = \frac{a_1}{A-1}, \tag{4.3}$$

$$\breve{y} = \frac{\alpha(k_0-\breve{x})a_1A}{k_0c_1(A-1)}.$$

where $A = \dfrac{p_1}{q_1 + \dfrac{c_2}{p_2}(p_2-q_2)}$.

Finally we obtained an interior equilibrium point $E^*(u^*,x^*,y^*,z^*)$ exists in the interior of the first orthant $(u>0, x>0, y>0, z>0)$ if there is positive solution to the following algebraic equations

$$\left(1-\frac{u^*}{L+lx^*}\right)=0,$$

$$\alpha\left(1-\frac{x^*}{K_0+ku^*}\right)-\frac{c_1y^*}{a_1+x^*+mu^*}=0, \tag{4.4}$$

$$\frac{p_1x^*}{a_1+x^*+mu^*}-q_1-\frac{c_2z^*}{y^*+z^*}=0, \qquad \frac{p_2y^*}{y^*+z^*}-q_2=0.$$

Thus, by solving above equations, we obtain

$$u^* = L + \frac{l(a_1+mL)}{A-(1+ml)},$$

$$x^* = \frac{a_1+mL}{A-(1+ml)},$$

$$y^* = \frac{\alpha}{c_1}\left(\frac{(k_0+kL)(A-(1+ml))+(a_1+mL)(kl-1)}{(k_0+kL)(A-(1+ml))+kl(a_1+mL))}\right)\left(\frac{(a_1+mL)A}{A-(1+ml)}\right). \tag{4.5}$$

$$z^* = \left(\frac{p_2 - q_2}{q_2}\right) y^*$$

Where, $A = \dfrac{p_1}{q_1 + \dfrac{c_2}{p_2}(p_2 - q_2)}$

It can be seen that $E^*(u^*, x^*, y^*, z^*)$ exist if and only if

(i) $p_2 > q_2$ (4.6)

(ii) $A > 1 + ml$

5. Local stability analysis

Now, in order to study the behaviour of solutions near equilibrium points, we need to compute variational matrix and characteristic equation of the system (2.1). The signs of the real parts of eigen values evaluated at a given equilibrium points determine the stability. The variational matrix for the system (2.1) is given by

$$V(u,x,y,z) = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix},$$

$$a_{11} = 1 - \frac{2u}{(L+lx)}, a_{12} = \frac{u^2 l}{(L+lx)^2}, a_{21} = \frac{c_1 mxy}{(a_1 + x + mu)^2} + \frac{\alpha x^2 k}{(k_0 + ku)^2},$$

$$a_{22} = \left(\alpha\left(1 - \frac{x}{k_0 + ku}\right) - \frac{c_1 y}{(a_1 + x + mu)}\right) - \frac{\alpha x}{k_0 + ku} + \frac{c_1 xy}{(a_1 + x + mu)^2}, a_{23} = -\frac{c_1 x}{(a_1 + x + mu)^2},$$

$$a_{31} = -\frac{mp_1 xy}{(a_1 + x + mu)^2}, a_{32} = \frac{yp_1(a_1 + mu)}{(a_1 + x + mu)^2}, a_{33} = \frac{p_1 x}{(a_1 + x + mu)} - q_1 - \frac{c_2 z}{y+z} + \frac{c_2 yz}{(y+z)^2},$$

$$a_{34} = -\frac{c_2 y^2}{(y+z)^2}, a_{43} = \frac{p_2 z^2}{(y+z)^2}, \quad a_{44} = -q_2 + \frac{p_2 y}{y+z} - \frac{p_2 yz}{(y+z)^2}.$$

Now we consider the various equilibrium states separately:

The variational matrix $V(\overline{E})$ at the equilibrium point $\overline{E}(0, \overline{x}, \overline{y}, 0)$ is given by

$$V(\bar{E})=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \dfrac{\alpha\bar{x}^2k}{k_0{}^2}+\dfrac{c_1m\,\bar{x}\,\bar{y}}{(a_1+\bar{x})^2} & \dfrac{-\alpha\bar{x}}{k_0}+\dfrac{c_1\bar{x}\,\bar{y}}{(a_1+\bar{x})^2} & \dfrac{-c_1\bar{x}}{(a_1+\bar{x})^2} & 0 \\ \dfrac{-mp_1\bar{y}\bar{x}}{(a_1+\bar{x})^2} & \dfrac{a_1p_1\bar{y}}{(a_1+\bar{x})^2} & 0 & -c_2 \\ 0 & 0 & 0 & p_2-q_2 \end{bmatrix}$$

Eigen values are positive in u and z directions. The other two eigen values are the roots of equation

$$\lambda^2-\left(-\frac{\alpha\bar{x}}{k_0}+\frac{c_1\bar{x}\,\bar{y}}{(a_1+\bar{x})^2}\right)\lambda+\frac{p_1a_1c_1\bar{x}\,\bar{y}}{(a_1+\bar{x})^3}=0$$

From Routh-Hurwitz criteria, the roots of equilibrium have negative real parts if and only if $-\dfrac{\alpha\bar{x}}{k_0}+\dfrac{c_1\bar{x}\,\bar{y}}{(a_1+\bar{x})^2}<0$. So $\bar{E}$ has nonempty unstable and stable manifold.

The variational matrix $V(\hat{E})$ evaluated at $\hat{E}(\hat{u},\hat{x},\hat{y},0)$ is written as

$$V(\hat{E})=\begin{bmatrix} -1 & l & 0 & 0 \\ \dfrac{\alpha\hat{x}^2k}{(k_0+k\hat{u})^2}+\dfrac{mc_1\hat{x}\,\hat{y}}{(a_1+\hat{x}+m\hat{u})^2} & \dfrac{-\alpha\hat{x}}{k_0+k\hat{u}}+\dfrac{c_1\hat{x}\,\hat{y}}{(a_1+\hat{x}+m\hat{u})^2} & \dfrac{-c_1\hat{x}}{a_1+\hat{x}+m\hat{u}} & 0 \\ \dfrac{-mp_1\hat{y}\hat{x}}{(a_1+\hat{x}+m\hat{u})^2} & \dfrac{p_1\hat{y}(a_1+m\hat{u})}{(a_1+\hat{x}+m\hat{u})^2} & 0 & -c_2 \\ 0 & 0 & 0 & p_2-q_2 \end{bmatrix}$$

The eigen value in the z direction is p_2-q_2, which is positive but other two eigen values are the zeros of the polynomial

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0, \tag{5.1}$$

Where

$$b_1=1+\frac{\alpha\hat{x}}{k_0+k\hat{u}}-\frac{c_1\hat{x}\,\hat{y}}{(a_1+\hat{x}+m\hat{u})^2},$$

$$b_2=\frac{p_1c_1\hat{x}\,\hat{y}(a_1+m\hat{u})}{(a_1+\hat{x}+m\hat{u})^3}+\frac{\alpha\hat{x}}{k_0+k\hat{u}}-\left[\frac{c_1\hat{x}\,\hat{y}}{(a_1+\hat{x}+m\hat{u})^2}+\frac{\alpha\hat{x}^2kl}{(k_0+k\hat{u})^2}+\frac{mc_1\hat{x}\,\hat{y}l}{(a_1+\hat{x}+m\hat{u})^2}\right],$$

$$b_3=\frac{p_1c_1\hat{x}\,\hat{y}a_1+p_1c_1\hat{x}\,\hat{y}m(\hat{u}-l)}{(a_1+\hat{x}+m\hat{u})^3}>0.$$

Thus from the Routh-Hurwitz criteria, the necessary and sufficient condition for $\hat{E}$ to be asymptotically stable in $u-x-y$ plane if $b_1>0, b_3>0$ and $b_1b_2-b_3>0$.

The variational matrix $V(\breve{E})$ for equilibrium point $\breve{E}(0,\breve{u},\breve{x},\breve{y},\breve{z})$ is given by

$$V(\breve{E})=\begin{bmatrix} 1 & 0 & 0 & 0 \\ \dfrac{\alpha\breve{x}^2k}{k_0{}^2}+\dfrac{c_1\breve{x}\breve{y}}{(a_1+\breve{x})^2} & \dfrac{-\alpha\breve{x}}{k_0}+\dfrac{c_1\breve{x}\,\breve{y}}{(a_1+\breve{x})^2} & \dfrac{-c_1\breve{x}}{a_1+\breve{x}} & 0 \\ \dfrac{-mp_1\breve{y}\breve{x}}{(a_1+\breve{x})^2} & \dfrac{a_1p_1\breve{y}}{(a_1+\breve{x})^2} & \dfrac{c_2\breve{y}\breve{z}}{(\breve{y}+\breve{z})^2} & \dfrac{-c_2\breve{y}^2}{(\breve{y}+\breve{z})^2} \\ 0 & 0 & \dfrac{p_2\breve{z}^2}{(\breve{y}+\breve{z})^2} & \dfrac{-p_2\breve{y}\breve{z}}{(\breve{y}+\breve{z})^2} \end{bmatrix}$$

The eigen value corresponding to u direction is 1 which is positive, gives unstable manifold, and the other three eigen values are the zeros of polynomial

$$\lambda^3+d_1\lambda^2+d_2\lambda+d_3=0, \tag{5.2}$$

$$d_1=\frac{\alpha\breve{x}}{k_0}+\frac{(p_2-c_2)\,\breve{y}\breve{z}}{(\breve{y}+\breve{z})^2}-\frac{c_1\breve{x}\,\breve{y}}{(a_1+\breve{x})^2}>0$$

$$d_2=\frac{(c_2-p_2)\,c_1\breve{x}\breve{y}^2\breve{z}}{(a_1+\breve{x})^2(\breve{y}+\breve{z})^2}+\frac{\alpha(p_2-c_2)\,\breve{x}\breve{y}\breve{z}}{k_0(\breve{y}+\breve{z})^2}+\frac{c_1p_1a_1\breve{x}\,\breve{y}}{(a_1+\breve{x})^3}$$

$$d_3=\frac{c_1p_1p_2a_1\breve{x}\,\breve{y}^2\breve{z}}{(a_1+\breve{x})^3(\breve{y}+\breve{z})^2}>0.$$

Thus from Routh-Hurwitz criteria will be asymptotically stable in R^+_{xyz} if and only if $d_1>0,\ \ d_3>0$ and $d_1d_2-d_3>0$.

Finally we investigate the local stability of interior equilibrium $E^*(u^*,x^*,y^*,z^*)$. However mutualistic interactions can have significant effect on stability, even in the complex plane. We first find the variational matrix $V(E^*)$ at interior equilibrium point E^* is

$$V(E^*)=\begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{bmatrix},$$

Where, $m_{11}=-1$, $m_{12}=l$, $m_{21}=\dfrac{\alpha x^{*2}k}{(k_0+ku^*)^2}+\dfrac{c_1x^*y^*m}{(a_1+x^*+mu^*)^2}$,

$$m_{22}=\frac{-\alpha x^*}{(k_0+ku^*)}+\frac{c_1x^*y^*}{(a_1+x^*+mu^*)^2},\ m_{23}=\frac{-c_1x^*}{a_1+x^*+mu^*},\ m_{31}=\frac{-p_1my^*x^*}{(a_1+x^*+mu^*)^2},$$

$$m_{32}=\frac{p_1(a_1+mu^*)y^*}{(a_1+x^*+mu^*)^2},\ m_{33}=\frac{c_2y^*z^*}{(y^*+z^*)^2}\ \ m_{34}=-\frac{c_2y^{*2}}{(y^*+z^*)^2},\ m_{43}=\frac{p_2z^{*2}}{(y^*+z^*)^2},$$

$$m_{44}=-\frac{p_2y^*z^*}{(y^*+z^*)^2}.$$

The eigenvalues of the above variational matrix are the roots of the characteristic polynomial

$$\lambda^4+A_1\lambda^3+A_2\lambda^2+A_3\lambda+A_4=0 \tag{5.3}$$

Where

$$A_1 = -(m_{11}+m_{22}+m_{33}+m_{44})'$$

$$A_1 = 1+\frac{\alpha x^*}{(k_0+ku^*)}\frac{(p_2-c_2)y^*z^*}{(y^*+z^*)^2}-\frac{c_1x^*y^*}{(a_1+x^*+mu^*)^2},$$

$$A_2 = m_{33}m_{44}+m_{11}m_{22}+m_{11}m_{33}+m_{11}m_{44}+m_{22}m_{44}+m_{22}m_{33}-m_{12}m_{21}-m_{23}m_{32}-m_{34}m_{43},$$

$$A_2 = \frac{\alpha x^*(p_2-c_2)y^*z^*}{(k_0+ku^*)(y^*+z^*)^2}+\frac{(p_2-c_2)y^*z^*}{(y^*+z^*)^2}+\frac{p_1c_1x^*y^*(a_1+mu^*)}{(a_1+x^*+mu^*)^3}+\frac{\alpha x^*(k_0+ku^*)-\alpha x^{*2}kl}{(k_0+ku^*)^2}$$
$$+\frac{c_1(c_2-p_2)x^*y^*z^*}{(a_1+x^*+mu^*)^2(y^*+z^*)^2}-\frac{c_1x^*y^*(1+m)}{(a_1+x^*+mu^*)^2},$$

$$A_3 = -m_{12}m_{23}m_{31}+m_{11}m_{23}m_{32}+m_{12}m_{21}m_{33}-m_{11}m_{22}m_{33}+m_{11}m_{34}m_{43}+m_{22}m_{34}m_{43}+m_{12}m_{21}m_{44}$$
$$-m_{11}m_{22}m_{44}+m_{23}m_{32}m_{44}-m_{11}m_{33}m_{44}-m_{22}m_{33}m_{44}$$
$$=\frac{p_1c_1x^*y^*(a_1+mu^*-mx^*)}{(a_1+x^*+mu^*)^3}+\frac{(p_2+c_2)c_1x^*y^{*2}z^*}{(a_1+x^*+mu^*)^2(y^*+z^*)^2}\frac{p_2p_1c_1x^*y^{*2}z^*(a_1+mu^*)}{(a_1+x^*+mu^*)^3(y^*+z^*)^2}$$
$$-\left[\frac{\alpha x^*y^*(p_2+c_2)}{(k_0+ku^*)(y^*+z^*)^2}+\frac{\alpha klx^{*2}y^*z^*(p_2-c_2)}{(k_0+ku^*)^2(y^*+z^*)^2}+\frac{mlc_1x^*y^{*2}z^*(p_2-c_2)}{(a_1+x^*+mu^*)^2(y^*+z^*)^2}\right]$$

$$A_4 = m_{12}m_{21}m_{34}m_{43}-m_{11}m_{22}m_{34}m_{43}+m_{12}m_{23}m_{31}m_{44}-m_{11}m_{23}m_{32}m_{44}+m_{11}m_{22}m_{33}m_{44}-m_{12}m_{21}m_{33}m_{44}$$

$$A_4 = \frac{p_1c_1p_2x^*y^*z^*((a_1+mu^*)y^*-mx^*l)}{(a_1+x^*+mu^*)^3(y^*+z^*)^2}>0.$$

Now in view of above calculation, we have following result.

Theorem (5.1): Equilibrium state is locally asymptotically stable if and only if
(i). $A_4>0,\quad A_2>0,\quad A_1>0$.
(ii). $A_3(A_1A_2-A_3)>A_1^2A_4$.

This theorem directly follows from the Routh-Hurwitz criterion if $(a_1+mu^*)y^*>mx^*l$. Hence E^* is locally asymptotically stable.

6. Persistence

From biological point of view, persistence means the survival of all populations of in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of non-negative cone. Butler et al [1986], Freedman & Waltman [1984&1985] developed following definition of persistence.

Definition: A population $N(t)$ is said to persist (sometimes called strongly persist) if $N(0)>0 \Rightarrow N(t)>0$ and $\liminf_{t\to\infty} N(t)>0$. Further, a population $N(t)$ is said to persist uniformly (also known as permanence) if $N(t)$ persists and there exist $\delta>0$ independent of $N(0)>0$, such that $\liminf_{t\to\infty} N(t)\geq\delta$. Finally, we say that a system persists (uniformly) whenever each component persists (uniformly).

Theorem (6.1): Assume that $\alpha a_1 \ell > c_1 M$ $p_2 > q_2$ and $q_1 + c_2 > B$ then system is permanent.

Where, $B = \dfrac{p_1 k_0 (\alpha a_1 \ell - c_1 M)}{\alpha a_1 \ell (a_1 + \tilde{k}(1+ml) + mL)}$

Proof: From the first equation of system (2.1), we have

$$\dot{u}(t) \ge u\left(1 - \frac{u}{L}\right)$$

Using comparison principle, we get

$$\liminf_{t\to\infty} u(t) \ge L(>0).$$

From the second equation of system (2.1), we have

$$\dot{x}(t) \ge \alpha x\left(1 - \frac{x}{k_0} - \frac{c_1 y_{\max}}{a_1}\right),$$

$$\ge \alpha x\left(1 - \frac{x}{k_0} - \frac{c_1 M}{a_1 \ell}\right),$$

Using comparison principle, we get

$$\liminf_{t\to\infty} x(t) \ge \frac{k_0(\alpha a_1 \ell - c_1 M)}{a_1 \alpha \ell}(>0).$$

This yield that $\alpha a_1 \ell > c_1 M$

$$\dot{y}(t) \ge \frac{p_1 x_{\min} y}{a_1 + x_{\max} + m u_{\max}} - q_1 y - \frac{c_2 y(t) z(t)}{y(t)+z(t)},$$

$$\ge \frac{p_1 k_0(\alpha a_1 \ell - c_1 M)}{a_1 \alpha \ell (a_1 + \tilde{k}(1+ml) + mL)} - q_1 y - c_2 y(t) + \frac{c_2 \ell y^2}{M}$$

According to comparison principle, it follows that

$$\liminf_{t\to\infty} y(t) \ge \frac{(q_1 + c_2 - B)M}{c_2 \ell}(>0).$$

This yield $q_1 + c_2 > B$ where $B = \dfrac{p_1 k_0(\alpha a_1 \ell - c_1 M)}{a_1 \alpha \ell (a_1 + \tilde{k}(1+ml) + mL)}$

From the fourth equation of system (2.1), we have

$$\dot{z}(t) \ge -q_2 z + p_2 z - \frac{p_2 c_2 \ell z^2}{(q_1 + c_2 - B)M},$$

By using comparison principle, we get

$$\liminf_{t\to\infty} z(t) \ge \frac{(p_2 - q_2)M(q_1 + c_2 - B)}{p_2 c_2 \ell}(>0).$$

This completes the proof of the theorem.

7. Numerical simulation

To facilitate the interpretation of our mathematical findings by numerical simulation, we integrate system (2.1) using fourth order Runga-kutta method under the following set of compatible parameters with the help of MATLAB Software package. Consider the following set of parameter values to study system (2.1), numerically.

$\alpha = 5$, $k_0 = 1.5$, $m = 0.1$, $c_1 = 4$, $c_2 = 0.7$, $p_1 = 2.3$, $p_2 = 3$, $L = 3,\ l = 0.24,\ a_1 = 1,\ k = 0.4,\ q_1 = 1,\ q_2 = 1.5$. (7.1)

For the above set of parameter values, we find that all the equilibrium point for the system exists and given by

$E_0(0,0,0,0)$, $E_1(3,0,0,0)$, $E_2(0,1.5,0,0)$ $\hat{E}(3.7168, 2.9867, 0,0)$, $\overline{E}(0,0.7692, 1.0774, 0)$, $\widetilde{E}(0,1.4211, 0.1593, 0.1593)$, $\breve{E}(3.2445, 1.0819, 1.8625, 0)$ and $E^*(3.4590, 1.9126, 1.3716, 1.3716)$

It is found that all the conditions of theorem (4.1) and theorem (6.1) for stability and permanence are satisfied and roots of characteristic equation becomes -2.13293, -0.81345, $-0.47811 \pm 4.03921i$. This shows that, unique positive equilibrium $E^*(u^*, x^*, y^*, z^*)$ is locally asymptotically stable.

Figures have been plotted between dependent variables and time for different parameter values to shows changes occurring in population with time under different conditions. For the existence, stability and persistence c_1, c_2, p_1 and p_2 are recognized to important parameter. The results of numerical simulation are displayed graphically, in figure (1) the mutualist, prey, first predator and top predator populations are plotted against time. From figure it is noted for given initial values both the populations tend to their corresponding value of equilibrium point E^* and hence coexist in the form of steady state assuring local stability of E^*.

Now if we set a parameters of the system (2.1) as, $\alpha = 5$, $k_0 = 1.5$, $m = 0.1$, $c_1 = 4$, $c_2 = 0.7$, $p_1 = 2.3$, $p_2 = 3$, $L = 3,\ l = 0.24,\ a_1 = 1,\ k = 0.4,\ q_1 = 1,\ q_2 = 3$.

Then we observe from the figure (2) that the top predator populations become extinct as a result population of first predator increases and prey population become decreases, consequently $\breve{E}(\breve{u}, \breve{x}, \breve{y}, 0) = (3.2445, 1.0819, 1.8625, 0)$ is locally asymptotically stable.

In figures 3 (a-d), variation of mutualist, prey, mature middle predator and top predator populations with time for different values of c_1 is shown. It is observed from the figure that mutualist population become stable and each population decreases with the increase in c_1. Which is obvious as prey population decreases, the middle predator and top predator population also decreases.

From figure 4(a-d), we conclude that y and z are decreasing function of c_2 , the capturing rate of top predators, whereas x and u is increasing function of c_2, which is obvious when capturing rate of top predator increases then the population of middle predator decreases and then population of prey increases and hence mutualist population slightly increases. Again if middle predator population decreases then population of top predator become decreases
.

From figure 5(a-d), we can depict that middle predator population decreases, mutualist, prey and top predator population increases with the increase in p_2, and finally attain their equilibrium levels. Which is obvious as the conversion rate for predation of middle predator by top predator are increases then the population of middle predator decreases and hence prey, mutualist and top predator population increases.

In figure 6(a-d), the variation of population with mutualist to prey, mutualist to middle predator, prey to middle predator and middle predator to top predator population is plotted for different initial starts. The graph depicts that solution converges to the equilibrium point E^* for different initial starts, indicating the global stability of E^*.

With the same set of parameters we found that when population of mutualist is absent, prey and both predator populations coexist in the form of stable equilibrium state. In figure (7), we see that in the presence of mutualist population, prey increases and hence population of predator also increases, but in the absence of mutualism, population of prey decreases and hence predator populations also decrease.

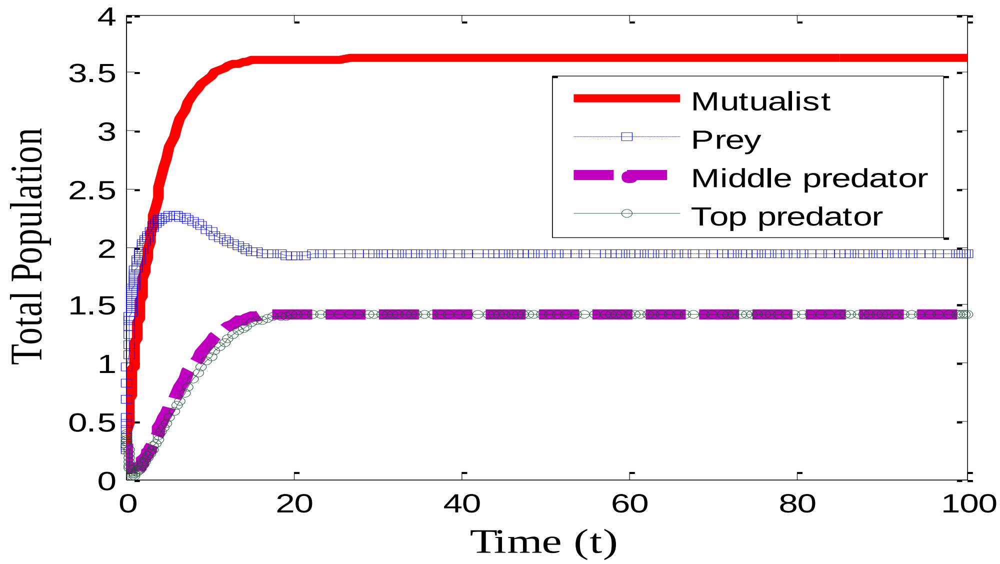


Fig. 1. Stable behaviour of u, x, y and z.

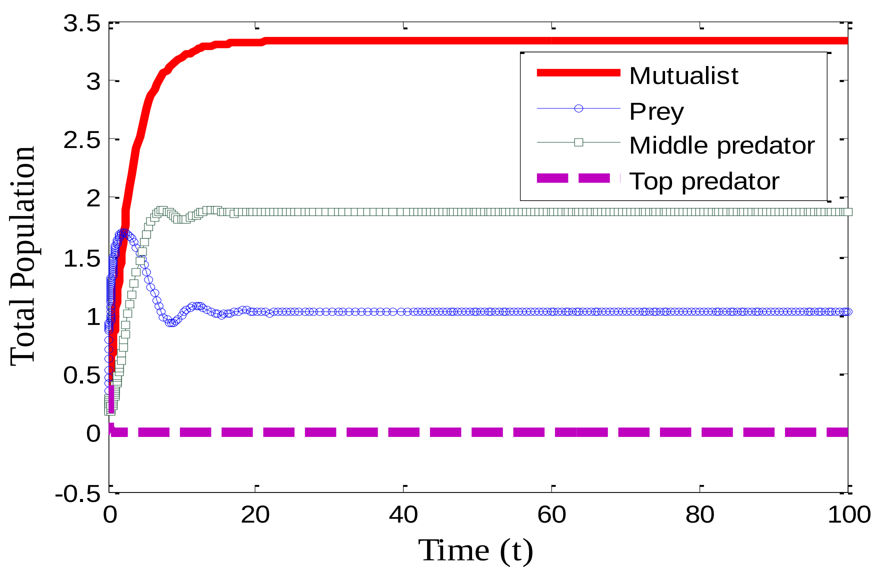


Fig. 2. Shows u, x, y populations approach to their equilibrium values in finite t and the $z-$population becomes extinct.

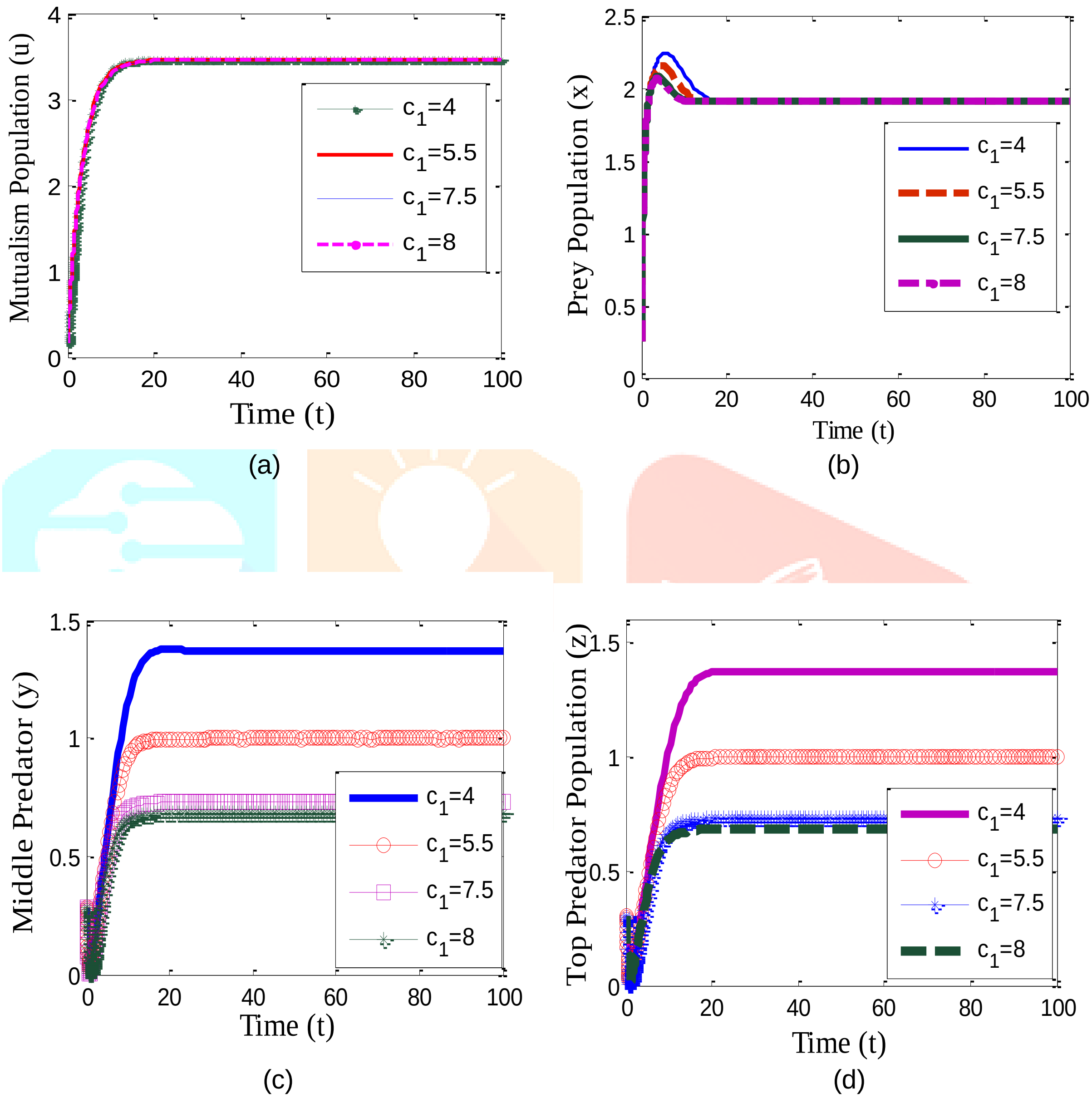


Fig. 3(a-d). Variation of mutualist, prey, middle predator and top predator populations with time for different values of c_1 and other parameter of values are same as in (7.1).

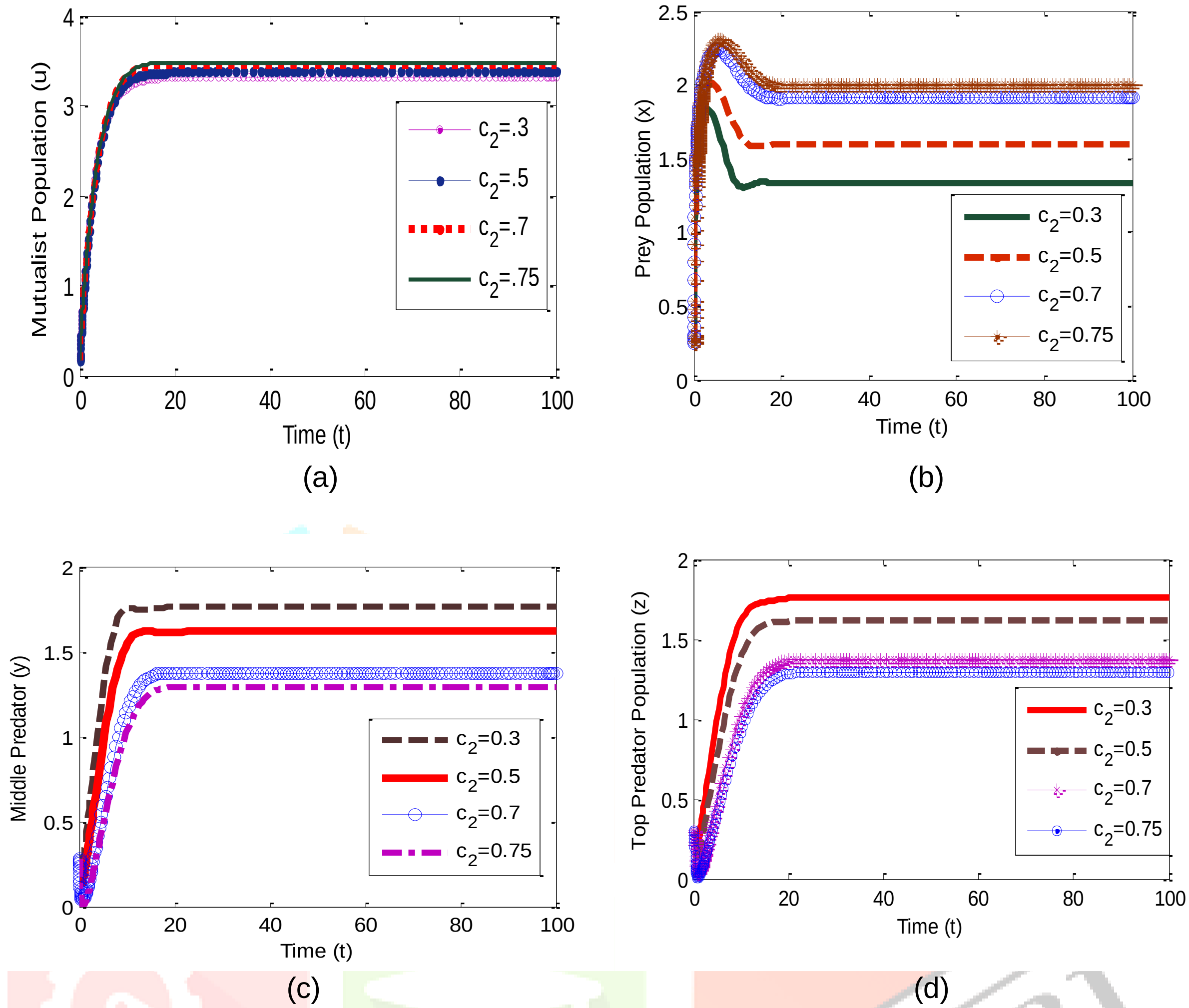
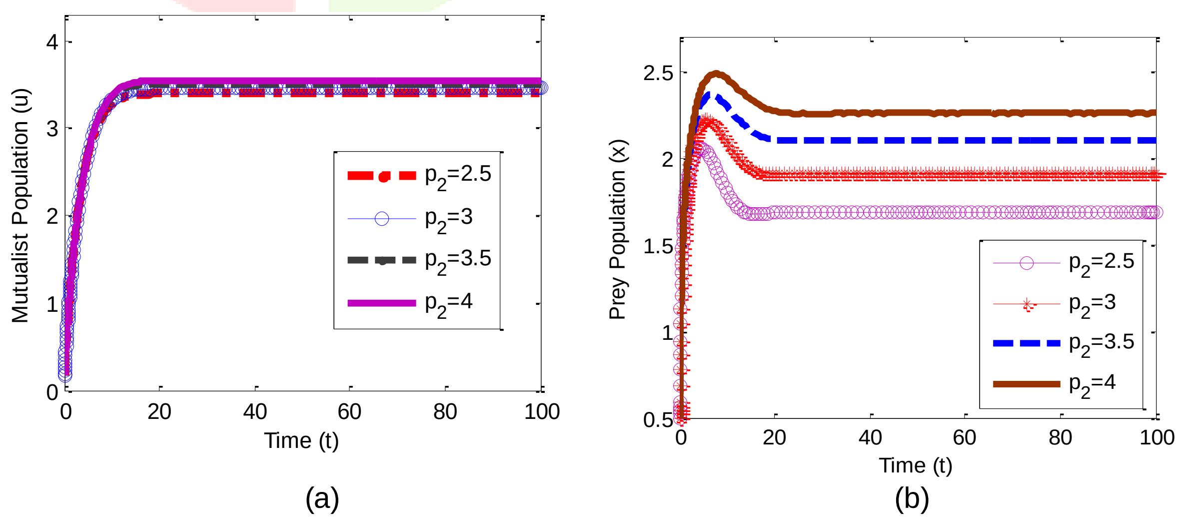
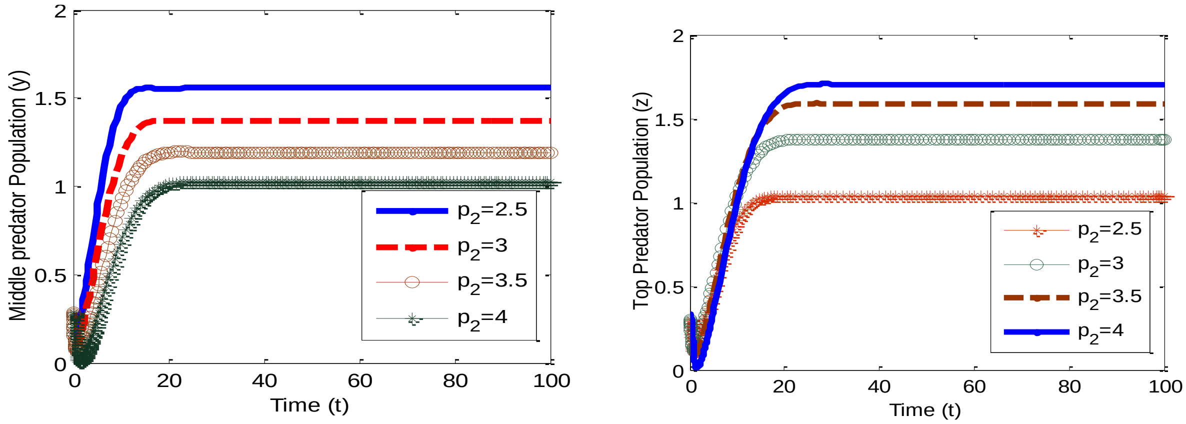


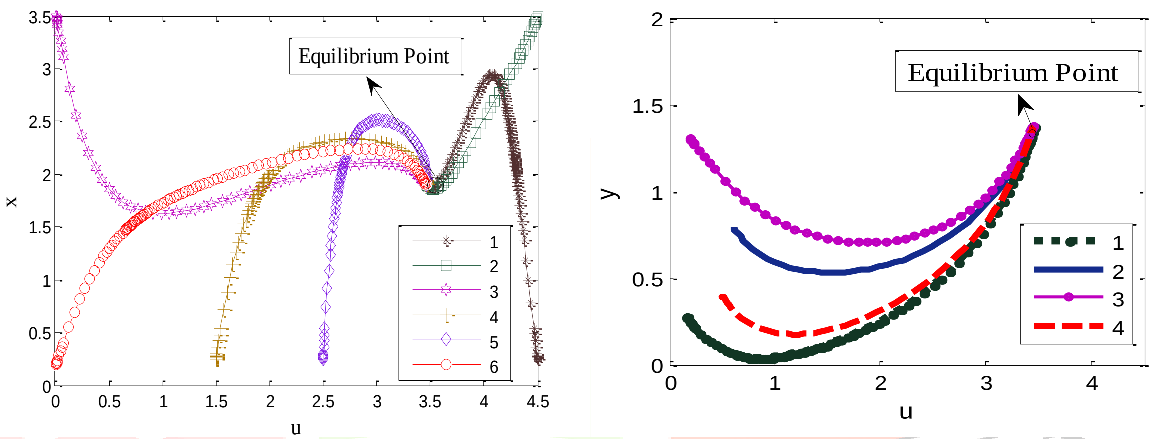
Fig. 4(a-d). Variation of mutualist, prey, middle predator and top predator populations with time for different values of c_2 and other parameter of values are same as in (7.1).



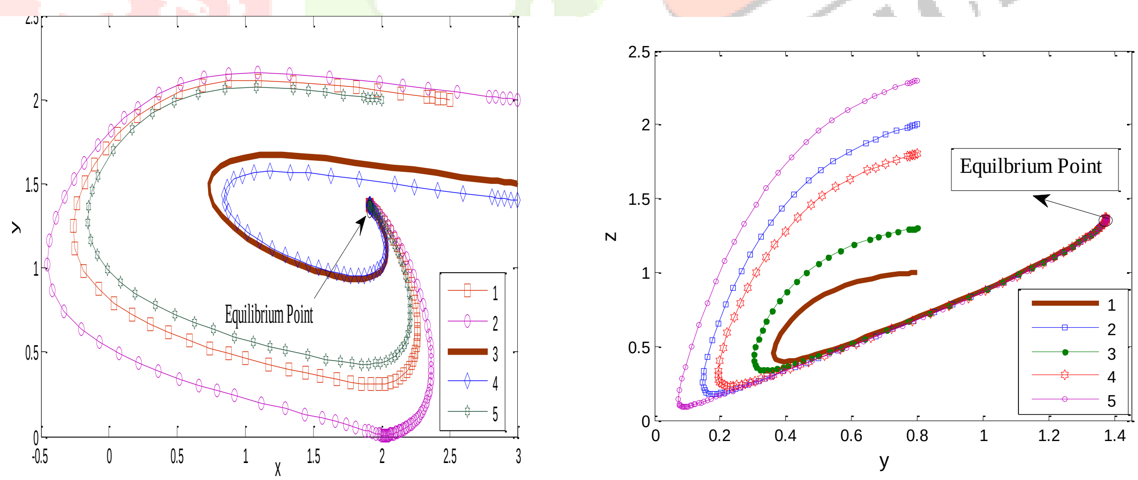


(c) (d)

Fig. 5(a-d). Variation of mutualist, prey, middle predator and top predator populations with time for different values of p_2 and other parameter of values are same as in (7.1)



(a) (b)



(c) (d)

Fig. 6(a-d). Variation of u with x, u with y, x with y and y with z for different intial starts for the set of parameters as same as given in (7.1).

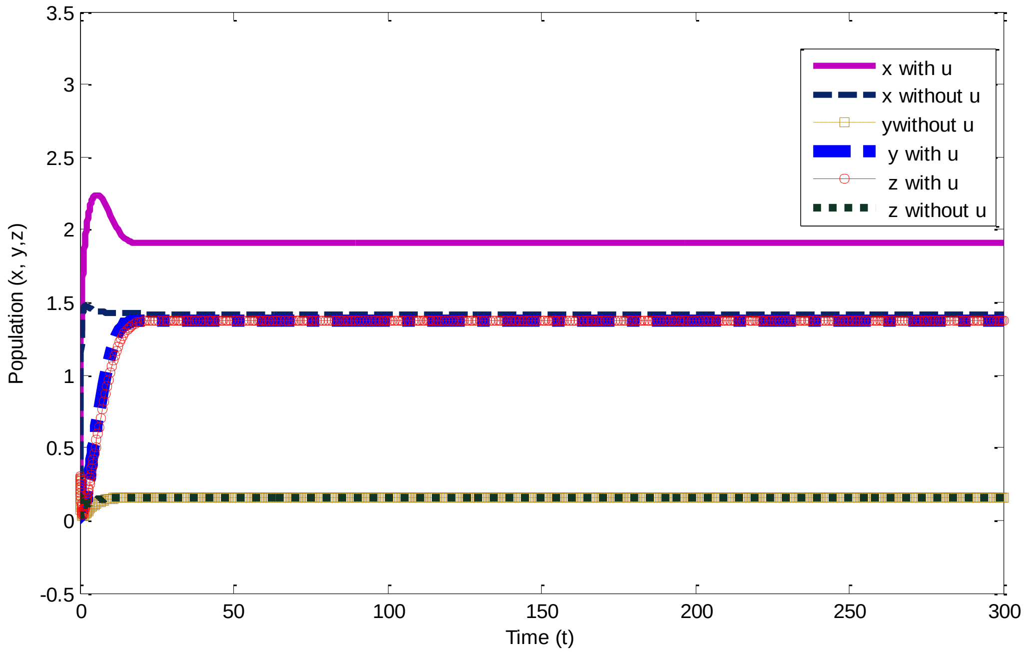


Fig.5. Comparison of populations (prey, middle predators and top predator) with and without mutualist population with the same set of parameters as (7.1).

8. Conclusion:

In this paper, the effect of mutualist on prey population on a tri-trophic food chain model with mixed selection of functional response is proposed and discussed. The above situation is described by means of a system with four nonlinear differential equation. The analysis consist of equilibrial stability and persistence criteria.

The system is analysed for boundedness of solutions, equilibria and their stabilities, existence conditions for equilibrium points of the system are determined. The stability and instability of the equilibrium points of the system are studied by using the linear stability approach. Criteria for long time survival (persistence of population) of system is interpreted biologically and obtained the condition which influence the persistence of all populations. $\alpha a_1 \ell > c_1 M$, $p_2 > q_2$ and $q_1 + c_2 > B$, these result indicates that the capturing rate, conversion rate and death rate of top predator plays an important role for the permanence of solutions.

With the help of computer simulation, it is observed that when the maximal growth rate of the top predator is equal to the death rate then top predator faces extinction. So we observed that when top predators are absent, predator y and prey x coexist and show stable behaviour. From figure (5), we have found that when mutualist population is present then population of prey increases and hence population of both predators also increases, but in the absence of mutualism, population of prey decreases and hence predator's population also decreases. The nonlinear differential equation (2.1) may be looked upon as the mathematical model for Acacia plant –ant–silky anteater Cyclopes didactyuls predator-eaters of ants such as –harpy eagle eats anteater etc. All our important mathematical findings and graphical representation of variety of solutions of system (2.1) are depicted by using MATLAB programming.

References:

1. Dean A. M. (1983). A simple Model of Mutualism. Am. Nat. 121, pp. 409-417.
2. Boucher D. N. (1995). The biology of mutualism, Croom Helm, London.
3. Addicott J. F. (1984). Mutualistic interactions in population and community processes. *John Wiley and Sons*, New York.
4. Addicott J. F. and Freedman H. I. (1984). On the structure and stability of mutualistic systems: Analysis of competition models as modified by a slow growing mutualist. Theor. Pop. Biol. 26, pp. 320-339.
5. Freedman H. I. (1987). Deterministic mathematical models in population ecology. HIFR Consulting Ltd.,Edmonton.
6. Kumar R. and Freedman H. I. (1985). A mathematical model of facultative mutualism with populations interacting in a food chain. Math. Biosci. 73, pp. 89-101.
7. Kumar R. and Freedman H. I. (2002). Mathematical analysis in a model of obligate mutualism with food chain populations. Nonlinear Dyn. and Sys.Theo. 2(1), pp. 25-44.
8. Freedman H. I. and Waltman P. (1977). Mathematical analysis of some three-species food-chain models, Math. Bioscicences 33, pp. 257–276.
9. Hsu SB, Hwang TW and Kuang Y (2003). A ratio-dependent food chain model and its applications to biological control, Mathematical Biosciences 181, pp. 55-83.
10. Freedman H. I., Kumar R., Easton A. K. and Singh M. (2001). Mathematical models of predator mutualism. Canad. Appl. Math. Qual. 9 No. 2, pp. 99-111.
11. Freedman H. I. and Rai B. (1995). Can mutualism alter competitive outcome? : A mathematical analysis. Rocky Mount. Jour. of Math. 25 (1) , pp. 217-230.