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Proper Roman Coloring of graphs

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Abstract:

Motivated from the traditional Roman military defense strategy, Suresh Kumar [12] introduced a new type of graph coloring called Roman coloring, and a related parameter called Roman Chromatic Number. However, it was not a proper coloring. In this paper, we introduce and study the Proper Roman colorings and the related Roman chromatic number.

Keyword: Graph, Proper Coloring, Roman coloring, Roman Chromatic Number.

1. Introduction

The coloring is also played an important role in combinatorial optimization problems and critical (Optimal) graphs were crucial in the Chromatic number Theory [7, 8, 9, 10, 11]. Jason Robert Lewis [1] suggested several new graph parameters in his Doctoral Thesis. Several studies were made in applying such parameters to Roman defense strategy [2, 3, 4, 5, 6]. The basic idea behind all these works was that for a given city, if the streets are considered as the edges of a graph and the meeting points of the streets, called the junctions, as the vertices of the graph, then we can color each vertex by the number of soldiers deployed at that junction and require that every street (edge) should be guarded by at least one soldier using a strategy that if any street have no soldier, then there must be an adjacent junction with two soldiers so that one among them may be deployed to the former junction in case of emergency.

Motivated from this Roman military defense strategy, Suresh Kumar [12] defined a new type of graph coloring, Roman Coloring and a related parameter, Roman Chromatic number. However, this is not a proper coloring. In this paper, we introduce and study the Roman colorings, which are colorings also. However, it is not a proper coloring. In this paper, we introduce and investigate the proper Roman coloring and the related Roman chromatic number denoted as $X_R(G)$. For the terms and definitions not explicitly here, refer Harary [13].

2. Main Results

Let G be a connected graph. A Proper Roman coloring of a graph G is an assignment of three colors, called {0, 1, 2}, to the vertices of G such that (1) adjacent vertices must have distinct colors and (2) any vertex with the color, 0 must be adjacent to a vertex with color, 2. The color classes will be denoted as V_0, V_1, V_2 which are the subsets of V(G) with colors 0, 1, 2 respectively.

Weight of a Roman coloring is defined as the sum of all the vertex colors. Proper Roman Chromatic number of a graph G is defined as the minimum weight of a Proper Roman coloring on G and is denoted by $X_R(G)$ A Proper Roman coloring of G with the minimal weight is called a minimal Proper Roman coloring of G. It follows from the definition of $X_R(G)$ that

Theorem 2.1. For any graph G, $X_R(G) \geq \chi(G)$.

Theorem 2.2. In a minimal Proper Roman coloring of a graph G, there is no edge joining V_1 and V_2

Proof. Consider a minimal Proper Roman coloring of a graph G. If possible, suppose that there is an edge e connecting a vertex $u \in V_1$ and a vertex $v \in V_2$. Since v is adjacent to u, this edge is incident to all edges incident at u. So we can change the color of u from one to zero, which is a contradiction to the minimality of the Proper Roman coloring of a graph G. Hence the theorem follows.

Theorem 2.3. There exists no graph with an edge and $X_R(G) = 1$,

Proof. The following result is immediate from the second condition of the Proper Roman coloring of a graph G that a vertex with the color, 0 must be adjacent to a vertex with color 2.

Theorem 2.4. $X_R(G) = 2$ if and only if $E(v) = E(G)$ for some vertex v of G, where $E(v) = \{e \in E(G): v \text{ is incident to } e\}$.

Proof. If $X_R(G) = 2$, then we have two cases to consider:

Case-1: There exists a vertex, v, with the color 2.

Since $X_R(G) = 2$, all other vertices of G are have the color, 0. Hence, $E(v) = E(G)$.

Case-2: There exists one edge whose end vertices have the color 1 each. Since $X_R(G) = 2$, no other vertex of G are have the color, 1 or 2. Also it cannot have the color 0. Hence the graph, G, must precisely be the graph with exactly one edge. Then $E(v) = E(G)$.

Conversely, if $E(v) = E(G)$, then v is incident with all edges of G, then the coloring of v with the color, 2 and coloring all other vertices of G with 0 is a Proper Roman coloring with $X_R(G)$= 2.

Theorem 2.5. For a graph G, $X_R(G) = 2$ if and only if G is a star graph

Proof. If $X_R(G) = 2$, then we have two cases to consider:

Case-1: There exists one edge whose end vertices have the color 1 each.

Then the graph must be precisely the graph with exactly this edge and is a star graph.

Case-2: There exists a vertex, v, with the color 2.

Since $X_R(G) = 2$, all edges of G are adjacent to v and have the color, 0. Then v is incident with all the edges of G so that G is a star graph.

Conversely, if G is a star graph, then, the coloring that assigns the color 2 to v and the color 0 to all other vertices of G is a Proper Roman coloring of G with $X_R(G)$=2.

Theorem 2.6. If a connected graph which is not a star graph and the minimum eccentricity and the maximum eccentricity of vertices in G are 1 and 2 respectively, then $X_R(G) = 2$.

Proof. Since G is a connected graph which is not a star graph and the minimum eccentricity and the maximum eccentricity of vertices in G are 1 and 2 respectively, there exists a vertex v, which is adjacent to all other vertices. So the Assignment of the color 2 to v and the color, 0 to all other vertices gives a Proper Roman coloring with $X_R(G)$=2.

Theorem 2.7. In a minimal Proper Roman Coloring of a graph G, if $X_R(G)$= 3, then exactly one vertex each with the colors 1 and 2.

Proof. In a minimal Proper Roman coloring of a graph G, $X_R(G) = 3$ is possible in two ways:

Case-1: There exists exactly two vertices u, v with the colors 1, 2 respectively.

Since $X_R(G)$= 3, al other vertices must have color, 0 and by the second condition of Proper Roman Coloring, all of them must be adjacent to v. This is the required condition in the theorem.

Case-2: There exist three vertices u, v, w all with the same color, 1.

We will find another vertex x and a vertex among u, v, w, say u, such that the new coloring that assigns the color 2 to x, the color 1 to u and the color 0 to all the remaining vertices is a Proper Roman coloring of G. Since G is connected, there exist a pair of vertices among u, v, w, say u, v, such that the induced subgraphs induced by the neighborhoods, N[u] and N[v] have one vertex, x, in common. Then we can assign the color 2 to this common vertex, the color 1 to the third vertex and the color 0 to all the remaining vertices which will give a new Proper Roman coloring of G in which there are exactly two vertices u, v with the colors 1, 2 respectively. Then, this case reduces to case-1 and hence the theorem follows.

Theorem.2.8. If $G = K_{m,n}, m \geq n$, then, $X_R(G) = n + 1$.

Proof. Let $\{A, B\}, |A| = m, |B| = n$ be the bipartion of $K_{m,n}$. Assign the color 0 to all the vertices in A. Assign the color 2 to one vertex of B and assign the color, 1 to all other vertices of B. Then we get a minimal Proper Roman coloring with $X_R(G) = 2 + (n - 1) = n + 1$.

Theorem.2.9. $X_R(C_{2k+1}) = 2k + 1$ and $X_R(C_{2k}) = 2k - 1$

Proof.

Case.1. If $n = 2k + 1$ is odd

Assign the vertices $\{v_1, v_2, \dots v_{2k+1}\}$ of C_n such that odd-suffixed vertices have the color 0 except the last one, which can be colored with, 1. Assign the color 2 to all the remaining vertices. Then it is a proper Roman coloring of $C_{2k+1}$and we cannot add more colors to 0's so that it is a minimal Proper Roman coloring of C_{2k+1}. Then, $X_R(C_{2k+1}) = 2\left(\frac{n-1}{2}\right) + 1 = 2k + 1$.

Case.2. If $n = 2k$ is even

Assign the vertices $\{v_1, v_2, \dots v_{2k}\}$ of C_n such that odd-suffixed vertices have the color 0 and assign the color 2 to all the remaining vertices except the last one, which must be colored with, 1.. Then it is a minimal Proper Roman coloring of C_{2k+1}.So, $X_R(C_{2k}) = 2\left\lfloor\frac{2k-1}{2}\right\rfloor + 1 = 2(k - 1) + 1 = 2k - 1$.

Theorem.2.10. $X_R(P_{2k+1}) = 2k, k \geq 1$ and $X_R(P_{2k}) = 2k - 1, k \geq 2$

Proof.

Case.1. If $n = 2k + 1$ is odd

Assign the vertices $\{v_1, v_2, \dots v_{2k+1}\}$ of P_n such that odd-suffixed vertices have the color 0 and all other vertices can be colored with the color 2. Then it is a proper minimal Roman coloring of P_{2k+1} and $X_R(P_{2k+1}) = 2\left(\frac{n-1}{2}\right) = 2k$.

Case.2. If $n = 2k$ is even

Assign the vertices $\{v_1, v_2, \dots v_{2k}\}$ of P_n such that odd-suffixed vertices have the color 0 and assign the color 2 to all the remaining vertices except the last one, which must be colored with, 1. Then it is a minimal Proper Roman coloring of P_{2k} and $X_R(P_{2k}) = 2\left\lfloor\frac{2k-1}{2}\right\rfloor + 1 = 2(k-1) + 1 = 2k - 1$.

References:

[1] J. R. Lewis, Vertex - Edge and Edge – Vertex Parameters in Graphs, Ph.D. Thesis submitted to the Graduate School of Clemson University.

[2] D. Ochmanek, Time to restructure U.S defense force, Issues in Science & Technology, Winter 1996.

[3] Pagourtzis, P. Penna, K. Schlude, K. Steinhofel, D. S. Tailor and P.Windmayer, "Server placements, Roman domination and other dominating set variants", 2nd IF1P International conference on theoretical Computer Science, Montreal (2002), 280 - 291.

[4] Windmayer, Server placements, Roman domination and other dominating set variants, 2nd IFIP International conference on theoretical Computer Science, Montreal (2002), 280 - 291.

[5] Petersen, Defending Roman empire, MathTreck, September 11 (2000), available at www.maa.org.

[6] J. Suresh Kumar and Satheesh E.N, Roman labeling of graphs and Application to Military Strategy, International Journal of Mathematics Trends and Technology, Vol.52 (2), Dec, 2017.

[7] J. Suresh Kumar, Graph Colouring Parameters-A Survey, International Journal for Research in Applied Science & Engineering Technology (IJRASET), Volume 7, Issue IV, Apr 2019.

[8] J. Suresh Kumar, Pseudo-Complete Color Critical Graphs, International Journal for Research in Applied Science & Engineering Technology (IJRASET), Volume 6, Issue I, January 2018.

[9] J. Suresh Kumar, Diameter and Travers ability of PAN Critical Graphs, International Journal of Mathematics Trends and Technology (IJMTT), Volume 52, Number 7, December 2017.

[10] J. Suresh Kumar, Degrees and Degree Sequences of PAN Critical Graphs, Int. J. Math. And Appl., 6(1-B)(2018), 1025-1028.

[11] J. Suresh Kumar, Low and High Vertices in Edge Critical Graphs, International Journal of Mathematics Trends and Technology (IJMTT), Volume 65 (4), April 2019.

[12] J. Suresh Kumar, Roman colouring of Graphs and Application to Military Strategy, International Journal for Research in Applied Science & Engineering Technology, Volume 8 (III), March 2020.

[13] Frank Harary, Graph Theory, Reading mass, 1969.