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# Hardy spaces on the disk and its applications

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## Abstract

In this paper , we discuss the Hardy Hilbert Space on the open disk with center origin and radius unity. We have proved that $H^2$ Space is isomorphic to proper subspace of $L^2$ Space which has various applications in Quantumm Mechanics.

**Keywords : Lebesgue , Parseval Identity , Separable , orthonormal**

## 1 Preliminaries

### 1.0.1 Definition (Inner Product Space)

An *inner product space* is a vector space $W$ (over field $K$ = R *or* C) with an inner product defined on it.

Here, an inner product is an function $<,>: W \times W \to K$ which satisfies the following properties:–

1. $<\alpha u + v, w> \;=\; \alpha <u, w> + <v, w>$
2. $\overline{<u, v>} \;=\; <v, u>$
3. $<u, u> \;\geq 0$
4. $<u, u> = 0 \;\Leftrightarrow u = 0$ ( **for all scalers** $\alpha \in K$ **and for all vectors** $u, v, w \in W$)

**Note1 Every inner product space is a normed spaces with the norm induced by the inner product is given by**

$$||u|| = \sqrt{<u, u>}$$

**Note2 An normed space** $(W, ||.||)$ **is said to be complete if each cauchy sequence converges in** $W$.

## 1.1 Hilbert Space

An Hilbert Space is defined as the complete inner product space.

**Example:-**

$$l^2 = \{(x_0, x_1, \ldots) : x_n \in \mathbb{C}, \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$$

i.e. all the elements of $l^2$ are the sequence of all the complex numbers that are square-summable. Inner product on $l^2$ is given by :-

$$<(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}> = \sum_{n=0}^{\infty} x_n \overline{y_n}$$ (it is an Hilbert sequence space)

### 1.2 Definition (Orthonormal sets and sequences)

An subset $X$ of an inner product space is said to be orthonormal if for all $u, v \in X$ we have ,

$$< u, v >= \begin{cases} 0 & \text{if} u \neq v \\ ||u||^2 & \text{if} u = v. \end{cases}$$

**Note If norm of each element of an orthogonal set $X$ is 1 then the set is said to be orthogonal. i.e for all $u, v \in X$ we have,**

$$< u, v >= \begin{cases} 0 & \text{if} u \neq v \\ 1 & \text{if} u = v \end{cases}$$

### 1.3 Definition (Orthonormal basis)

An orthonormal subset $X$ of Hilbert space $W$ is said to be an orthonormal basis if span of $X$ is dense in $W$. i.e.

$$\overline{Span\ X} = W$$

**Note**Every Hilbert space $W$ not equals to {0} has an orthonormal basis.

### 1.4 Definition (Separable Hilbert Space)

A Hilbert-Space $W$ is said to be *separable* if there exist a countable set which is dense in $W$.

**Example:** $l^2$ is a separable Hilbert space

**Note** Each orthonormal basis of an separable Hilbert space are countable. Therefore orthonormal basis of $l^2$ are countable

**Recall**

1. An orthonormal sequence $(e_n)_{n=0}^{\infty}$ is an orthonormal basis of a Hilbert - Space $W$

$$\Leftrightarrow$$

for all $u \in W$ we have $\sum_{n=0}^{\infty} |< u, e_n >|^2 = ||u||^2$ [2] **Parseval identity**

2. Let $(e_n)$ be an orthonormal sequence in a Hilbert-space then

$$\sum_{n=0}^{\infty} \alpha_n e_n$$

converges in $W$ *iff*

the series

$$\sum_{n=0}^{\infty} |\alpha_n|^2$$

converges in R

## 2 THE HARDY-HILBERT SPACE

### 2.1 DEFINITION

It is defined as the space of all the analytic functions which have a power series representation about origin with square-summable complex coefficients. It is denoted by $H^2$.

$$H^2 = \{f : f(z) = \sum_{n=0}^{\infty} \alpha_n z^n : \sum_{n=0}^{\infty} |\alpha_n|^2 < \infty\}$$

Inner Product on $H^2$ is given by

$$< f,g >= \sum_{n=0}^{\infty} a_n \overline{b_n}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n \quad and \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad in \quad H^2$

**Theorem 2.1.** *The Hardy-Hilbert space is a separable Hilbert Space.*

*Proof.* Define an function;–

$$\Phi : l^2 \to H^2$$

given by

$$(a_n)_{n=0}^{\infty} \to \sum_{n=0}^{\infty} a_n z^n$$

- **$\varphi$ is well defined** since $(a_n)_{n=0}^{\infty} \in l^2 \Rightarrow \sum_{n=0}^{\infty} |a_n|^2 < \infty \Rightarrow \sum_{n=0}^{\infty} a_n z^n$ which being an power series is an analytic function whose coefficients are square summable hence is in $H^2 \therefore \varphi$ is well defined

- **Clearly $\varphi$ is linear**

- **$\varphi$ is isometric**

  Fix $(a_n)_{n=0}^{\infty} \in l^2$ then we have

$$\phi((a_n)_{n=0}^{\infty}) = ||\sum_{n=0}^{\infty} a_n z^n||_{H^2} = \sqrt{\sum_{n=0}^{\infty} |a_n|^2} = ||(a_n)_{n=0}^{\infty}||_{l^2}$$

  $\therefore \varphi$ is an isometric

  $\therefore \varphi$ preserves the norm so that the inner product

- **since isometry property implies one one property**

  $\therefore \varphi$ **is one one** [1]

- **$\varphi$ is onto**

  Let $f \in H^2$ then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$

  **define** $x = (a_0, a_1, ...)$ **Since**

$$||x||^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

$$\therefore x \in l^2$$

  **and**

$$\varphi(x) = f$$

$$\therefore \varphi \quad is \quad onto$$

**Therefore $\varphi$ is an vector space isomorphism which also preserves the inner product.**
**Since $l^2$ is an separable Hilbert space hence $H^2$ is also an separable Hilbert Space**

□

**Notations** D = $\{z : |z| < 1\}$ denotes the open unit disk about origin in C $S^1 = \{z : |z| = 1\}$ denotes the unit circle about origin in C

**Theorem 2.2.** *Radius of convergence of each function in $H^2$ is atleast* 1 *(i.e. each function in $H^2$ is analytic in the open unit disk* D*)*

*Proof.* Let $z_0 \in$ D is fixed ⇒ $|z_0| < 1$ ∴ the geometric series $\sum_{n=0}^{\infty} |z_0|^n$ converges. Let $f \in H^2$ is arbitrary. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

Since the series $\sum_{n=0}^{\infty} |a_n|^2$ converges ⇒ $|a_n|^2 \dashrightarrow 0 \Rightarrow |a_n| \dashrightarrow 0$
∴ $(|a_n|)_{n=0}^{\infty}$ is an convergent sequence hence bounded. ∴ $\exists M > 0$ such that

$$|a_n| \leq M \qquad \forall \qquad n \geq 0$$

Now

$$\sum_{n=0}^{\infty} |a_n z_0^n| \leq M \sum_{n=0}^{\infty} |z_0|^n$$

where being an geometric series right hand side converges.
∴ By Comparison test the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges absolutely . Since in Hilbert space absolute convergence implies convergence.
∴ the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges in $H^2$ since $z_0 \in H^2$ is arbitrary ∴ each function in $H^2$ is analytic in the unit disk D

□

## 2.2 Definition ($L^2(S^1)$ space)

It is defined as the space of all the equivalence classes of functions [4] that are Lebesgue measurable on $S^1$ and square integrable on $S^1$ with respect to Lebesgue measure normalized such that measure of $S_1$ is 1.

$$L^2(S^1) = \{f : f \text{ is Lesbesgue measurable on } S^1 \text{ and } \frac{1}{2\pi}\int_0^{2\pi} |f(e^{\iota\theta})|^2 d\theta < \infty\}$$

Inner product on $L^2(S^1)$ is given by -

$$< f, g > = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\iota\theta})\overline{g(e^{\iota\theta})} d\theta$$

**Note** $L^2(S^1)$ is an Hilbert-space with the orthonormal basis given by $\{e_n : n \in Z\}$ where $e_n(e^{\iota\theta}) = e^{\iota n\theta}$.

**Therefore**

$$L^2(S^1) = \left\{ f : f = \sum_{n=-\infty}^{n=\infty} < f, e_n > e_n \right\} \text{..[3]}$$

### 2.2.1 Definition ($H_C^2$ space)

$$H_C^2 = \{f \in L^2(S^1) : < f, e_n > = 0 \text{ for negative value of } n\}$$

$$H_C^2 = \left\{ f \in L^2(S^1) : f = \sum_{n=0}^{\infty} < f, e_n > e_n \right\}.$$

**$H_C^2$ is an subspace of $L^2(S^1)$ whose negative Fourier coefficients are 0**

$\therefore \{e_n : n = 0,1,...\}$ **are orthonormal basis of** $\widehat{H^2}$ **Theorem 2.3.** $\overline{H_C^2}$ *is an Hilbert-space*

*Proof.* Let $f \in \overline{H_C^2}$ then there exist an sequence $(f_n)_{n=0}^{\infty}$
such that $f_n \longrightarrow f$ as $n \longrightarrow \infty$
Since $f_n \in H_C^2$ $\forall$ $n \geq 0$

$\therefore$ $< f_n, e_k >= 0$ $\forall n \geq 0$ *and* $\forall k < 0$

**Now for each** $k < 0$ **we have**

$| < f_n, e_k > - < f, e_k > | \leq | < (f_n - f, e_k > | \leq ||f_n - f|| \longrightarrow 0$ *as* $n \longrightarrow \infty$(Schwarz Inequality [2])
since $< f_n, e_k >= 0$ $\forall n \geq 0$ $\Rightarrow$ $< f, e_k >= 0$
Since $k < 0$ is arbitrary $\therefore$ $< f, e_k >= 0$ $\forall$ $k < 0$

$\therefore$ $f \in H_C^2$

**Therefore** $\overline{H_C^2}$ **is an closed subspace of** $L^2(S^1)$ **Hence an Hilbert-Space**

□

**Theorem 2.4.** *The Hardy-Hilbert space can be identified as a subspace of* $L^2(S^1)$

*Proof.* Define an function

$$\psi : H^2 \to \widehat{H^2}$$

$$f \to \tilde{f}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\tilde{f} = \sum_{n=0}^{\infty} a_n e_n$

- **$\psi$ is well defined**

  Let $f \in H^2$ Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$
  Then by **(recall 2)** the series $\tilde{f} = \sum_{n=0}^{\infty} a_n e_n$ converges in $H^2$
  $\therefore \psi$ is well defined
- **Clearly $\psi$ is linear**
- **$\psi$ is an isometry**

  For any arbitrary $f \in H^2$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have:-

$$||\psi(f)|| = ||\tilde{f}|| = \frac{1}{2\pi}\int_0^{2\pi} |\tilde{f}(e^{\iota\theta}|^2 d\theta$$

**Now**

$$\frac{1}{2\pi}\int_0^{2\pi} |\tilde{f}(e^{\iota\theta})|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi} (\sum_{n=0}^{\infty} a_n e^{\iota n\theta})(\overline{\sum_{m=0}^{\infty} a_m e^{\iota m\theta}})$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\overline{a_m}e^{\iota(n-m)\theta}d\theta$$

$$= \sum_{n=0}^{\infty}|a_n|^2 \qquad \textbf{(since } \frac{1}{2\pi}\int_0^{2\pi} e^{\iota(n-m)\theta} = \delta_{nm})$$

$$= ||f||^2$$

Since $f \in H^2$ is arbitrary

$$\therefore ||\psi(f)|| = ||f|| \quad \forall \quad f \in H^2$$

**Therefore $\psi$ is an isometry. Hence it preserves the inner product Isometry $\Rightarrow$ one one property. $\therefore \psi$ is one one.**

- **$\psi$ is Onto**

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Let $\tilde{f} \in \widehat{H^2}$. Then $\tilde{f} = \sum_{n=0}^{\infty} < f, e_n > e_n$

where $< f,e_1 >, < f,e_2 >, \ldots$ are Fourier coefficients of f with respect to the orthonormal basis $\{e_n : n \in \mathbb{N}\}$.

**Then by Parseval relation we have**

$$\sum_{n=0}^{\infty} | < f, e_n > |^2 = ||f||^2 < \infty$$

**Define**

$$f = \sum_{n=0}^{\infty} a_n z^n \qquad \text{where} \qquad a_n = < f,e_n > \qquad \forall \qquad n \geq 0$$

**Since**

$$\sum_{n=0}^{\infty} |a_n|^2 = \sum_{n=0}^{\infty} | < f,e_n > |^2 = ||f||^2 < \infty$$

**Therefore** $f \in H^2$

**That is for each $\tilde{f} \in \widehat{H^2}$ there exist $f \in H^2$ such that $\psi(f) = \tilde{f}$**

**Therefore $\psi$ is onto**

**That is $\psi$ is a vector space isomorphism which also preserves the norm. Therefore $H^2$ can be identified as a subspace of the $L^2(S^1)$ space**

□

# 3 Applications

1. In the mathematical rigrous formulation of Quantum Mechanics, developed by **Joh Von Neumann**' the position and momentum states for a single non relavistic spin 0 Particle is the space of all the square integrable functions($L^2$). But $L^2$ have some undesirable properties and $H^2$ is much well behaved space so we work with $H^2$ instead of $L^2$.

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