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ON WEAKER FORM CONNECTEDNESS WITH RESPECT TO AN IDEAL

[1]R.THENMOZHY [2]R.ALAGAR [3]C.JESURAJ

[1]Assistant Professor in Mathematics, Periyar Arts College –Cuddalore
[2]Assistant Professor, Rajeshwari vedhachalam government arts college –Chengalpattu
[3]Associate Professor in Mathematics ,IFET College of engineering-Villupuram

Abstract: An ideal on a set X is a nonempty collection of subsets of X with heredity property which is also closed under finite unions. In this article we introduce the concept of ideal-connected spaces using ideals, called $\mathfrak{I}$ - connected spaces and extend some important results on connectedness to $\mathfrak{I}$ - connectedness. Also, we introduce the concept of Strongly ideal-connected spaces using ideals, called Strongly $\mathfrak{I}$ - connected spaces and extend some important results on Strongly connectedness to $\mathfrak{I}$ - connectedness. Conditions on $\mathfrak{I}$ are obtained under which connectedness, $\mathfrak{I}$ - connectedness and strongly $\mathfrak{I}$ - connectedness are equivalent.

Keywords: **$\mathfrak{I}$-connected, * - closure, ideal component, strongly $\mathfrak{I}$ - connectedness.**

AMS Subject Classification No: *54A05,54A10,54C08*

1.INTRODUCTION

In1945, R. Vaidyanathaswamy [13] introduced the concept of ideal topological spaces. T. R. Hamlett, D. A. Rose [4] defined the local function and studied some topological properties using local function in ideal topological spaces in 1990. Since then many mathematicians studied various topological concepts in ideal topological spaces. The first unified and extensive study on τ * - topologies was done by Jankovic and Hamlett in [2] and proofs for the facts stated above may be found in [6]. The initial important articles on topological spaces are [4] and [5], a thesis [3] and a book that includes ideal is [12].. In this article we introduce the concept of ideal-connected spaces using ideals, called $\mathfrak{I}$ - connected spaces and extend some important results on connectedness to $\mathfrak{I}$ - connectedness.

2. PRELIMINARIES

Given a nonempty set X, a collection $\mathfrak{I}$ of subsets of X is called an ideal if,

(i) $A\in\mathfrak{I}$ and $B\subseteq A$ implies $B\in\mathfrak{I}$ (heredity)

(ii) $A\in\mathfrak{I}$ and $B\in\mathfrak{I}$ implies $A\cup B\in\mathfrak{I}$ (additivity)

If $X\notin\mathfrak{I}$, then $\mathfrak{I}$ is called a proper ideal. An ideal $\mathfrak{I}$ is called a σ - ideal if the following holds:

If $\{A_n : n = 1,2, \ldots\}$ is a countable sub collection of $\mathfrak{I}$, then $\cup\{A_n : n = 1,2, \ldots\}\in\mathfrak{I}$

The notation $(X,\tau, \mathfrak{I})$ denotes a nonempty set X, a topology τ on X and an ideal $\mathfrak{I}$ on X. Given a point $x \in X$, $\aleph(x)$ denotes the neighbourhood system of x; that is, $\aleph(x) = \{U\in\tau : x\in U\}$. $\wp(X)$ denotes the collection of all subsets of X. Given space $(X, \tau, \mathfrak{I})$ and a subset A of X, we define

$A^* (\mathfrak{I},\tau) = \{x \in X : U \cap A \notin \mathfrak{I}, \text{ for every } U\in\aleph(x)\}$

We simply write A^* for $A^*(\mathfrak{I}, \tau)$, when there is only one ideal $\mathfrak{I}$ and only one topology τ under consideration. If we define cl^* on $\wp(X)$ as, $cl^*(A) = A \cup A^*$, for all $A \in \wp(X)$,

then cl^* is a Kuratowski closure operator. The topology determined by this closure operator is denoted by $\tau^*(\mathfrak{I})$. $\beta(\mathfrak{I}, \tau) = \{U - I : U \in \tau, I \in \mathfrak{I}\}$ is a basis for $\tau^*(\mathfrak{I})$. For every subset A of a given topological space $(X, \tau, \mathfrak{I})$, the sets cl(A) (or $\overline{A}$) and $cl^*(A)$ will denote closure of A with respect to τ and τ^* respectively.

3. $\mathfrak{I}$ - CONNECTED SPACES

Let us start with a definition for $\mathfrak{I}$ - connected spaces.

Definition: 3.1 Let (X, τ) be a topological space with an ideal $\mathfrak{I}$ on X. A subset of Y of X is said to be $\mathfrak{I}$-connected if $Y \neq A \cup B$, A, $B \notin \mathfrak{I}$ such that $\overline{A} \cap B = \phi = A \cap \overline{B}$

Remark :3.1 Every connected set is $\mathfrak{I}$ - connected. We give the following example to show that the converse need not be true.

Example :3.1 Let (R,τ) denote the real line with the usual topology and $\mathfrak{I}$ denote the ideal of all finite subsets of X. Let $Y = [0,2] \cup \{3,4,5\}$. Then Y is $\mathfrak{I}$- connected but not connected.

Remark:3.2 Let (X, τ) be a topological space with an ideal $\mathfrak{I}$ on X. Let X be $\mathfrak{I}$- connected. If $\mathcal{J}$ is an ideal on X with $\mathfrak{I} \subseteq \mathcal{J}$, then X is $\mathcal{J}$ - connected.

We obtain equivalent conditions for a space to be a $\mathfrak{I}$ - connected space in the following theorem.

Theorem: 3.2 Let (X,τ) be a topological space. Then the followings are equivalent.

(i) X is $\mathfrak{I}$- connected

(ii) X cannot be expressed as a union of two disjoint non-ideal open sets.

(iii) X cannot be expressed as a union of two disjoint non-ideal closed sets.

Proof: (i) $\Rightarrow$ (ii)

Suppose (ii) is not true, then $X = A \cup B$, for some subset $A, B \notin \mathfrak{I}$ such that A, B are open and $A \cap B = \varphi$. Then $A = \overline{A}$ and $B = \overline{B}$ so that $\overline{B} \cap A = \varphi = \overline{A} \cap B$ This contradicts (i). Therefore (ii) is true

(ii) $\Rightarrow$ (iii) : Suppose (iii) is false. Then $X = A \cup B$, for some subsets $A,B \notin \mathfrak{I}$ such that A,B are closed and $A \cap B = \varphi$. Then $X = A \cup B$, where $A, B \notin \mathfrak{I}$, $A \cap B = \varphi$ and $A = X - B$, $B = X - A$ are open. This contradicts (ii). Therefore (iii) is true.

(iii) $\Rightarrow$ (i) : Suppose X is not $\mathfrak{I}$- connected. Then $X = A \cup B$, for some subsets $A, B \notin \mathfrak{I}$, such that $\overline{A} \cap B = \varphi = A \cap \overline{B}$. Then $\overline{A} \subseteq A$ and $\overline{B} \subseteq B$. Hence $X = A \cup B$, where $A, B \notin \mathfrak{I}$, $A \cap B = \varphi$ and A, B are closed; which is a contradiction to (iii). So X is $\mathfrak{I}$ - connected.

Remark :3.3 Let (X, τ) be a topological space and $\mathfrak{I}$ be an ideal on X. A subset Y of X is $\mathfrak{I}$ - connected if and only if it is not possible to find open sets A and B in X such that

(i) $Y \subseteq A \cup B$

(ii) $Y \cap A \notin \mathfrak{I}$, $Y \cap B \notin \mathfrak{I}$

(iii) $Y \cap \overline{A} \cap B = \phi$

(iv) $Y \cap A \cap \overline{B} = \phi$

We know that if $\{A_\alpha : \alpha \in \wedge\}$ is a collection of connected subsets of a space (X, τ) such that $\cap A_\alpha \neq \phi$, then $\cup A_\alpha$ is also connected in (X, τ) . Can this result be extended to a $\mathfrak{I}$ - connectedness?.We first have the following theorem to get a partial affirmative answer. However, we shall find an example that gives a negative answer to this questions.

Theorem: 3.3 Let A_1 and A_2 be two $\mathfrak{I}$-connected sets with $A_1 \cap A_2 \notin \mathfrak{I}$. Then $A_1 \cup A_2$ is $\mathfrak{I}$ - connected.

Proof : Suppose $A_1 \cup A_2$ is not $\Im$-connected. Then $A_1 \cup A_2 = C \cup D$ where $C, D \notin \Im$ and $(A_1 \cup A_2) \cap \overline{C} \cap D = \phi = C \cap \overline{D} \cap (A_1 \cup A_2)$.We have $A_1 \cap A_2 =$

$(C \cap A_1 \cap A_2) \cup (D \cap A_1 \cap A_2) \notin \Im$, So either $C \cap A_1 \cap A_2 \notin \Im$, or $D \cap A_1 \cap A_2 \notin \Im$. Suppose $C \cap A_1 \cap A_2 \notin \Im$, then $C \cap A_1 \notin \Im$ and $C \cap A_2 \notin \Im$. Since $A_1 = (C \cap A_1) \cup (D \cap A_1)$ is $\Im$- connected, either $C \cap A_1 \in \Im$ or $D \cap A_1 \in \Im$. As $C \cap A_1 \notin \Im$, we have $D \cap A_1 \in \Im$. Similarly, we have $D \cap A_2 \in \Im$

So $D = (D \cap A_1) \cup (D \cap A_2) \in \Im$, which is a contradiction. Hence $A_1 \cup A_2$ is $\Im$- connected.

Corollary: 3.4 The finite union of $\Im$- connected sets $\{A_1, A_2, \ldots, A_n\}$ for which $\bigcap_{i=1}^{n} A_i$ is a non-ideal set, is also an $\Im$- connected set.

But arbitrary union of $\Im$ - connected sets $\{A_i\}$, whose intersection $\bigcap_{i=1}^{\infty} A_i$ is a non-ideal set need not be $\Im$-connected. The following example justifies this statement.

Example: 3.2 Let X be the real line with the usual topology τ. Let $A_n = (0,1) \cup \{n+1\}$, for all n = 1,2… and let $\Im$ be the ideal of all finite subsets of X. Then A_n is an $\Im$ - connected. Also $\bigcap_{i=1}^{\infty} A_n$ is a non-ideal set. However, $\bigcup_{n=1}^{\infty} A_n = (0,1) \cup \{2,3,\ldots\}$ is not $\Im$ - connected.

Theorem: 3.5 Let (X,τ) be a topological space with an ideal $\Im$ on X. If $A \subseteq X$ is $\Im$- connected and $A \subseteq B \subseteq cl^*(A)$ (closure of A in τ^*), then B is $\Im$ - connected.

Proof : Suppose B is not $\Im$-connected. Then $B = C \cup D$, where $C, D \notin \Im$ and $B \cap \overline{C} \cap D = \varphi = C \cap \overline{D} \cap B$. Now $A = (A \cap C) \cup (A \cap D)$. Since A is $\Im$-connected, either $A \cap C \in \Im$ or $A \cap D \in \Im$. Suppose $A \cap D \in \Im$ and let $x \in D - A$. Then for every neighbourhood V of x, $V \cap A \notin \Im$. As $V \cap A = (V \cap A \cap C) \cup (V \cap A \cap D) \notin \Im$, we have $V \cap A \cap C \notin \Im$. In particular $V \cap A \cap C \neq \varphi \Rightarrow V \cap C \neq \varphi \Rightarrow x \in \overline{C}$. Therefore $x \in D - A \Rightarrow x \in \overline{C}$, which is contradiction to $B \cap \overline{C} \cap D = \varphi$. Hence $D - A = \varphi$ i.e., $D \subseteq A$. Therefore $D = D \cap A \in \Im$, which is a contradiction. Thus B is $\Im$ - connected.

The above theorem is not true, if we replace * - closure with closure. We give the following example to justify this fact.

Example :3.3 Let X be the real line with the usual topology. Let $A = [0,1] \cup \{x : x \text{ is rational}, 4 < x < 5\}$ and let $\Im$ be the ideal of zero measurable sets. Then A is $\Im$ - connected, but $\overline{A} = cl(A) = [0,1] \cup [4,5]$ is not $\Im$ - connected. We know that the continuous image of connected set is connected. We generalize this in the following theorem.

Theorem: 3.6 Let $f: (X, \tau, \Im) \to (Y, \sigma)$ is a continuous surjection . If (X,τ) is $\Im$ - connected, then (Y,σ) is $f(\Im)$ - connected, where $f(\Im) = \{f(I) : I \in \Im\}$

Proof : Let $f : (X, \sigma, \Im) \to (Y,\sigma)$ is a continuous surjection map and X is $\Im$ - connected. Assume that Y is not $f(\Im)$ - connected, then $Y = B \cup C$, where $B,C \notin f(\Im)$, $B \cap C = \varphi$ and B, C are open.

Since f is continuous, $f^{-1}(B)$, $f^{-1}(C)$ are open and $f^{-1}(B) \cap f^{-1}(C) = f^{-1}(B \cap C) = f^{-1}(\phi) = \phi$. Also $f^{-1}(B), f^{-1}(C) \notin \Im$ (if $f^{-1}(B) \in \Im$, then $B \in f(\Im)$, gives contradiction). Now $X = f^{-1}(B) \cup f^{-1}(C)$, where $f^{-1}(B), f^{-1}(C)$ are open, $f^{-1}(B) \cap f^{-1}(B) = \phi$ and $f^{-1}(B)$, $f^{-1}(C) \notin \Im$. Hence X is not $\Im$ - connected; a contradiction to our assumption. Thus Y is $f(\Im)$ - connected.

In the following lemma, we show that extensions of $\Im$-connected spaces by members of $\Im$ are $\Im$- connected.

Lemma : 3.7 Let (X, τ) be a topological space with an ideal $\Im$ on X. Let $A, B \subseteq X$. If A is $\Im$- connected and $B \in \Im$ then $A \cup B$ is $\Im$ - connected. (In particular, Let (X, τ) be a topological space with an ideal $\Im$ on X. Let $A \subseteq X$. If A is $\Im$- connected and $X - A \in \Im$, then X is $\Im$ - connected).

Proof : If $A \cup B$ is not $\Im$ - connected, then there exist open sets C and D in X such that $A \cup B = C \cup D$ and

(i) $(A \cup B) \cap C \notin \Im$, $(A \cup B) \cap D \notin \Im$.

(ii) $(A \cup B) \cap (\overline{C} \cap D) = \phi$, $(A \cup B) \cap (C \cap \overline{D}) = \phi$

As $B \in \Im$, we have $A \cap C \notin \Im$ (as $B \cap C \in \Im$) and $A \cap D \notin \Im$.

As $A = (A \cap C) \cup (A \cap D)$, which is a contradiction to $\Im$- connectedness of A. Hence $A \cup B$ is $\Im$ - connected.

We have already given example for $\Im$- connected spaces which are not connected. But if the ideal $\Im$ satisfies some extra conditions, then we can expect that the space is connected if only if it is $\Im$- connected. In the following theorem we show that if the ideal is a τ - boundary ideal i.e. $\Im \cap \tau = \{\phi\}$, then concept of connectedness and $\Im$ - connectedness coincide.

Theorem: 3.8 Let (X, τ) be a topological space with an ideal $\Im$ on X. If X is $\Im$-connected and $\Im \cap \tau = \{\varphi\}$, then X is connected.

Proof: Suppose X is not connected, then $X = A \cup B$, where $A \neq \varphi$, $B \neq \varphi$ and $\overline{A} \cap B = \phi = A \cap \overline{B}$. Since $\Im \cap \tau = \{\varphi\}$, we have A, $B \notin \Im$. So, X is not $\Im$- connected, a contradiction. Thus X is connected

Theorem : 3.9 Let (X, τ) be an $\Im_1$- connected space and let (Y, σ) be an $\Im_2$- connected space. Assume that $\Im_1 \cap \tau$ is closed under arbitrary unions. If $\Im$ is an ideal such that $P_i^{-1}(\Im_i) \subset \Im$, i = 1,2, then X x Y is $\Im$- connected.

Proof : If $X \in \Im_1$, then X x Y is in the ideal $\Im$ and hence X x Y is $\Im$- connected so assume that $X \notin \Im_1$.

Assume that X x Y is not $\Im$ - connected , then $X \times Y = A \cup B$, where A, $B \notin \Im$, $A \cap B = \varphi$ and A, B are open in X x Y.

To each $y \in Y$, define $A_y = \{x \in X : (x,y) \in A\}$ and $B_y = \{x \in X : (x,y) \in B\}$

Let $C = \{y \in Y : A_y \in \Im_1 \}$ and $D = \{y \in Y : B_y \in \Im_1 \}$

Then $X = A_y \cup B_y$. To each y , both A_y and B_y are open and $A_y \cap B_y = \varphi$. As X is $\Im_1$ - connected , either $A_y \in \Im_1$ or $B_y \in \Im_1$. In fact, to each $y \in Y$, exactly one of A_y and B_y belongs to $\Im_1$.

Therefore $Y = C \cup D$ and $C \cap D = \varphi$. Now we claim that C is closed. Fix $y \in \overline{C}$. If $A_y \notin \Im_1$, then $A_y \neq \varphi$. Since A is open, to each $x \in A_y$, there exist neighbourhoods U_x of x and V_y of y such that $(x, y) \in U_x \times V_y \subset A$. As $y \in \overline{C}$, there is one $y' \in V_y \cap C$, so $U_x \times \{y'\} \subseteq A$ and hence $U_x \subset A_{y'}$ and as $A_{y'} \in \Im_1$, we have $U_x \in \Im_1$. Therefore $A_y \subseteq \cup \{ U_x : x \in A_y \} \in \Im_1$(by assumption).

Hence $A_y \in \Im_1$ and hence $y \in C$. Therefore C is closed. Similarly D is closed. Since Y is $\Im_2$ - connected, we have $C \in \Im_2$ or $D \in \Im_2$

Case (i): If $C \in \Im_2$, then $X \times C \in \Im$. Take $E = \cup \{B_y : y \in D \} \in \Im_1 \cap \tau$ (assumption), So $E \times Y \in \Im$ and $(X \times C) \cup (E \times Y) \in \Im$. Fix $(x, y) \in B$. If $y \in C$, then $(x, y) \in X \times C$. If $y \notin C$, then $y \in D$ and $x \in B_y \subseteq E$. Therefore $(x,y) \in E \times Y$. Hence $B \subseteq (X \times C) \cup (E \times Y)$, So $B \in \Im$, This contradicts the fact $B \notin \Im$.

Case (ii:) If $D \in \Im_2$,then $X \times D \in \Im$; and as in case (i), we obtain a contradiction. Thus X x Y is $\Im$-connected.

Corollary : 3.10 Let ($X_i \tau_i$) be topological spaces with ideals $\Im_i$ on X_i respectively for i = 1,2,….n . Let $X = \overset{n}{\underset{i=1}{\pi}} \pi$ X_i ; and $\Im$ be an ideal such that $P_i^{-1} (\Im_i) \subset \Im$, i = 1,2…..n.

If $\{\Im_i \cap \tau_i$: i = 1,2….n-1}is closed under arbitrary unions and X_i is $\Im_i$ - connected, then X is $\Im$ - connected.

Corollary: 3.11 Let (X, τ) be a connected space and (Y,σ) be $\Im_2$ - connected. If $\Im$ be an ideal containing $P_2^{-1} (\Im_2)$, then X x Y is $\Im$ - connected.

Proof: Consider $\Im_1 = \{\varphi\}$, then $\tau \cap \Im_1$ is closed under arbitrary unions, so by theorem 2.9, X x Y is $\Im$-connected.

Definition: 3.2 Let (X, τ) be a topological space and $\Im$ be an ideal in X. A connected component C of X with respect to τ is called τ - connected component of X . A τ - connected component C of X is said to be an ideal component in X if $C \in \Im$.

Example : 3.4 Let X = { [0,1] ∪ [2,3] ∪[4,5] ∪ …… [2n, 2n+1] ∪ ……..} with the subspace topology induced by the usual topology of R. If $\Im$ is the set of all bounded subsets, then every component of X is an ideal component.

Theorem : 3.10 Let (X, τ_1) be $\Im_1$ - connected and (Y, τ_2) be $\Im_2$ - connected. Assume that any union of ideal components is a member of $\Im_1$. If $\Im$ is an ideal in X x Y containing $P_1^{-1} (\Im_1)$ and $P_2^{-1}(\Im_2)$, then X x Y is $\Im$ - connected.

Proof: If $X \in \Im_1$ then X x Y is in the ideal $\Im$ and hence X x Y is $\Im$ - connected. So we assume that $X \notin \Im_1$. Assume that X x Y is not $\Im$ - connected . Then $X \times Y = A \cup B$, where A, $B \notin \Im$, $A \cap B = \phi$ and A,B are open sets. For every component C of X and D of Y,C x D is a connected subset of X x Y and hence $C \times D \subseteq A$ or $C \times D \subseteq B$……………(1)

For every component D of Y, write

$A_D = \cup \{ C : C$ is a component of X and $C \times D \subseteq A \}$.

$B_D = \cup \{ C : C$ is a component of X and $C \times D \subseteq B \}$.

Now we claim that A_D is open. Let $x \in A_D$, then there exists a component C of X such that $x \in C$ and $C \times D \subseteq A$. Fix $y \in D$. Therefore $(x,y) \in C \times D \subseteq A$. Since A is open, there exist neighbourhoods U_x , V_y of x, y respectively such that $U_x \times V_y \subseteq A$. If $x \in cl(B_D)$, then $U_x \cap B_D \neq \phi$. Let $x' \in U_x \cap B_D$ i.e. $x' \in U_x \cap C_0$, for some component C_0 where $C_0 \times D \subseteq B$. Let $(x', y) \in U_{x'} \times V_y' \subseteq B$, where $U_{x'}$, V_y' are some neighbourhoods of x′, y respectively. Then $(x', y) \in (U_x \cap U_{x'}) \times (V_y \cap V_y') \subseteq A \cap B$, which contradicts $A \cap B = \phi$. Therefore $x \in A_D$ implies that x is not a limit point of B_D . That is, A_D is open. Similarly B_D is open. Thus $X = A_D \cup B_D$, and A_D B_D are open. So exactly one of A_D, B_D is in $\Im_1$, because $X \notin \Im_1$.

Le $\mathcal{D}_1 = \{ D \subseteq Y : D$ is component of Y and $A_D \in \Im_1\}$ and

$\mathcal{D}_2 = \{ D \subseteq Y : D$ is component of Y and $B_D \in \Im_1\}$.

Write $D_1 = \bigcup_{D \in \mathcal{D}1} D$ and $D_2 = \bigcup_{D \in \mathcal{D}2} D$ Then $Y = D_1 \cup D_2$ and $D_1 \cap D_2 = \phi$

We claim that D_1 is closed . Fix $d \in \overline{D}_1$. Let D be the component of Y such that $d \in D$. Suppose $d \notin D_1$. Then $D \in \mathcal{D}_1 \Rightarrow A_D \notin \Im_1$ so $B_D \in \Im_1$. By (1) and our assumption, there is a component C of X such that $C \notin \Im_1$, $C \times D \subseteq A$. Fix a member $c \in C$. Then $(c,d) \in A$. Since A is open, there exist neighborhoods U_c , V_d of c, d in X, Y respectively, such that $(c, d) \in U_c \times V_d \subseteq A$. So there is a member $d' \in V_d \cap D_1$ and there is a component D′ of Y such that $d' \in D'$ and $A_{D'} \in \Im_1$, so that $(c, d') \in (U_c \times V_d) \subseteq A$. Therefore $(C \times D') \subseteq A$ and $C \in \Im_1$, because $C \subseteq A_{D'} \in \Im_1$. This contradicts $C \notin \Im_1$.Therefore $d \in D_1$. i.e. D_1 is closed. Similarly D_2 is closed. Thus $Y = D_1 \cup D_2$, where D_1, D_2 are closed and $D_1 \cap D_2 = \phi$. Since Y is $\Im_2$ - connected, either $D_1 \in \Im_2$ or $D_2 \in \Im_2$. Without loss of generality, we assume that $D_1 \in \Im_2$ .Then $X \times D_1 \in \Im$. Take Take $E = \bigcup_{D \in D_2} B_D \in \tau$ (by assumption). So $E \times Y \in \Im$ and hence $(X \times D_1) \cup (E \times Y) \in \Im$. It is enough to prove that $B \subseteq (X \times D_1) \cup (E \times Y)$. Fix $(x, y) \in B$. Then there exist components C and D such that $(x, y) \in C \times D \subseteq B$. If $y \in D_1$, then $(x, y) \in X \times D_1$. If $y \notin D_1$ then Take $y \in D_2 = \bigcup_{D \in D_2} D$ and hence $x \in C \subseteq B_D \subseteq E$, for some $D \in \mathcal{D}_2$. Therefore $(x,y) \in E \times Y$. Hence $B \subseteq (X \times D_1) \cup (E \times Y) \in \Im$. This is a contradiction to $B \notin \Im$. Hence $X \times Y$ is $\Im$ - connected.

4. STRONGLY $\Im$ - CONNECTED SETS

Let us begin with following definition.

Definition: 4.1 Let (X,τ) be a topological space and let $\Im$ be a ideal on X. A subset A of X is said to be strongly $\Im$ - connected if there is a τ - connected subset B of X such that $A = B \cup C$, where $C \in \Im$.

Every connected set is strongly $\Im$ - connected set, but converse need not be true. It follows from the following example.

Example: 4.1 Let (R, τ) denote the set of real numbers with the usual topology and $\Im$ be the ideal of all finite subsets of R. Let $A = [0,2] \cup \{3,4,5\}$. Then A is strongly $\Im$ - connected, but not connected.

The following theorem gives the relation between $\Im$ - connectedness and strongly $\Im$ - connectedness.

Theorem: 4.1 Let (X, τ) be a topological space with a ideal $\Im$ on X. If (X, τ) is strongly $\Im$ - connected, then it is $\Im$- connected.

Proof: Assume that $(X, \Im)$ is strongly $\Im$ - connected and $X = B \cup C$ where B is τ - connected and $C \in \Im$. Suppose $X = D_1 \cup D_2$, where D_1 , D_2 are open and $D_1 \cap D_2 = \phi$. Then $B = (D_1 \cap B) \cup (D_2 \cap B)$ and $D_1 \cap B = \phi$ or $D_2 \cap B = \phi \Rightarrow D_1 \subset X - B$ or $D_2 \subset X - B \Rightarrow D_1 \subset C$ or $D_2 \subset C \Rightarrow D_1 \in \Im$ or $D_2 \in \Im$. Hence X is $\Im$ - connected.

The converse of the above theorem is not true.

Example: 4.2 Let $X = \{0, 1, 1/2, 1/3, \ldots\}$ and τ be the topology denoted by the usual topology in R. Let $\Im$ be the ideal of all finite subsets. Then X is $\Im$ - connected. For if X is not $\Im$ - connected, then $X = B \cup C$, where B, $C \notin \Im$, $B \cap C = \phi$ and B, C are open.

Therefore $0 \in B$ or $0 \in C$, which implies that C is finite or B is finite, so that $C \in \Im$ or $B \in \Im$ which is a contradiction. But X is not strongly $\Im$ - connected because only connected subsets of X are singletons whose its compliments are not in $\Im$.

Theorem: 4.2 Given (X, τ, ℑ) such that the ideal ℑ is a τ - boundary ideal. Then the following are equivalent:

(i) X is connected

(ii) X is ℑ- connected

(iii) X is strongly ℑ- connected

Remark:4.1 Let (X,τ) be a topological space with a ideal ℑ on X. Let A, B ⊆ X. If A is strongly ℑ- connected and B ∈ℑ, then A ∪ B is strongly ℑ- connected.

Theorem: 4.3 Let A_i (i = 1,2……n) be strongly ℑ- connected sets such that $\bigcap_{i=1}^{n} A_i \notin \Im$, then $\bigcup_{i=1}^{n} A_i$ is strongly ℑ- connected.

Proof: Since each A_i (i = 1,2,…n) is strongly ℑ- connected, we have $A_i = B_i \cup C_i$, where B_i is connected and $C_i \in \Im$. As $\cap A_i \notin \Im$, we get $A_i \notin \Im$, and $B_j \notin \Im$, for all j. Let $F_j = (\cap A_i) \cap C_j$. Then $F_j \in \Im$, for all j.

Therefore $\bigcup_{i=1}^{n} F_j \in \Im$,. Put E = $(\cap A_i) - (\cup F_j)$. Then $E \notin \Im$, because $\cap A_i \notin \Im$

and $F_j \in \Im$. Now $E \subset B_j$ for all j and hence $E \subset \cap B_j \notin \Im$. In particular $\cap B_j \neq \phi$

Hence $\bigcup_{i=1}^{n} B_j$ is connected and hence $\bigcup_{i=1}^{n} A_i = (\bigcup_{i=1}^{n} B_j) \cup C$, where

$C \subseteq \bigcup_{i=1}^{n} (A_i - B_i) \subseteq \bigcup_{i=1}^{n} C_i \in \Im$. Thus $\bigcup_{i=1}^{n} A_i$ is strongly ℑ - connected.

This theorem need not be true, if the family $\{A_i\}$ is an infinite family whose intersection is a non ideal set, as this may be seen from the following example.

Example: 4.3 Let X be the real line with the usual topology τ. Let $A_n = (0,1) \cup \{n+1\}$ for all n = 1,2… and let ℑ be the ideal of all finite subsets of X. Then A_n is strongly ℑ - connected.

Also $\bigcap_{n=1}^{\infty} A_n$ is a non ideal set. However, $\bigcup_{n=1}^{\infty} \cup A_n = (0,1) \cup \{2,3,\ldots..\}$ is not strongly ℑ - connected.

It is well known that the continuous image of a connected set is connected. This result can be generalized as follows.

Theorem: 4.4 Let f: (X, τ, ℑ) → (Y, σ) be a continuous surjection. If (X, τ) is strongly ℑ- connected, then (Y,σ) is strongly f (ℑ) – connected, where f (ℑ) = {f (I): I ∈ℑ }.

Proof : Let f: (X, τ, ℑ) → (Y, σ) be a continuous surjection and let (X, τ) is strongly ℑ- connected. Then X = B ∪ C, where B is connected and C ∈ℑ. Therefore Y = f (X) = f (B ∪ C) = f (B) ∪ f(C), where f (B) is connected and f (C) ∈ f (ℑ). Thus (Y, σ) is strongly f (ℑ) - connected.

Theorem: 4.5 If $\overline{I} \in \Im$, for all I ∈ℑ then whenever A is strongly ℑ- connected, then B is also strongly ℑ- connected, for all B with $A \subseteq B \subseteq \overline{A}$. In particular $\overline{A}$ is strongly ℑ- connected, if $\overline{I} \in \Im$, for all I ∈ℑ.

Proof: Suppose A is strongly ℑ- connected. Then A = C ∪ D, where C is connected and D ∈ℑ. Since $A \subseteq B \subseteq \overline{A}$ and A = C ∪ D ⊆ B, we have $B = (\overline{C} \cap B) \cup (\overline{D} \cap B)$, where $\overline{C} \cap B$ is connected as $C \subseteq \overline{C} \cap B \subseteq \overline{C}$ and $\overline{D} \cap B \in \Im$. Hence B is strongly ℑ- connected. As a particular case when A is strongly ℑ- connected, $\overline{A}$ is strongly ℑ- connected, for all $\overline{I} \in \Im$.

The condition of I ∈ℑ implies $\overline{I} \in \Im$ can not be relaxed from the previous theorem 4.5. This is justified by the next example.

Example: 4.4 Let (R,τ) denote the real line with the usual topology and let $\Im$ be the ideal of all with zero measure. Let $A = [0,1] \cup \{x : x$ is rational, $4 < x < 5\}$. Then A is strongly $\Im$ - connected, but $\overline{A} = [0,1] \cup [4,5]$ is not strongly $\Im$ - connected.

Example :4.5 Let $X = [0,1] \cup \{2,3,4,5\}$ with the usual topology and let $\Im$ be the ideal of all finite subsets of X. Then X is strongly $\Im$- connected, but X x X is not strongly $\Im \times \Im$ - connected.

Now we discuss strong- ideal connectedness of product of two strongly $\Im$- connected sets with a suitable ideal in the product space.

Theorem: 4.6 Let (X, τ_1) be strongly $\Im_1$- connected and (Y, τ_2) be strongly $\Im_2$ - connected. If $\Im$ is a ideal on X x Y such that $p_i^{-1}(\Im_i) \subset \Im$, i = 1,2 , then X x Y is strongly $\Im$ - connected, where $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are the projections and $p_i^{-1}(\Im_i) = \{ p_i^{-1}(I_i): I_i \in \Im_i, i=1,2.\}$

Proof: Suppose X is strongly $\Im_1$- connected and Y is strongly $\Im_2$ - connected. Then $X = A \cup C_1$ and $Y = B \cup C_2$, where A,B are connected subsets of X and Y respectively and $C_1, C_2 \in \Im$.

Then $X \times Y = (A \times B) \cup [(C_1 \times Y) \cup (X \times C_2)]$. Since A x B is connected with respect to the product topology $\tau_1 x \tau_2$ and $C_1 \times Y$, $X \times C_2 \in \Im$, we have $(C_1 \times Y) \cup (X \times C_2) \in \Im$. Thus X x Y is strongly $\Im$ - connected.

Corollary 4.7 Let (X_i ,τ_i), i = 1,2…n be a topological space with a ideal $\Im_i$ on X_i, for i = 1,2,3,....,n. If each X_i, i = 1,2….n is strongly $\Im_i$- connected and if $\Im$ is a ideal containing $p_i^{-1}(\Im_i)$, then $\underset{i=1}{\overset{n}{\pi}} X_i$ is strongly $\Im$ - connected, where $p_i : \underset{j=1}{\overset{n}{\pi}} X_j \to X_i$ are the projection and $p_i^{-1}(\Im_i) = \{ p_i^{-1}(I_i): I_i \in \Im_i , i=1,2,3,......n.\}$

Corollary: 4.8 Let (X ,τ_1) and (Y,τ_2) be two topological spaces with ideals $\Im_1$, $\Im_2$ on X, Y respectively. Let (X ,τ_1) be a strongly $\Im_1$ - connected and (Y,τ_2) be $\Im_2$ - connected. If $\Im$ is a ideal containing $p_1^{-1}(\Im_1)$ and $p_2^{-1}(\Im_2)$, then X x Y is $\Im$ - connected.

Consider the following definition.

Definition: 4.2 Let (X ,τ) be a topological space with a ideal $\Im$ on X. A subset $A \subseteq X$ is said to be $\Im$ - well linked if $\overline{A}$ is strongly $\Im$ - connected

From the theorem 2.9, it follows that if $\overline{I} \in \Im$, for all $I \in \Im$, then every strongly $\Im$- connected subsets of X are $\Im$ - well linked. The converse of the observation is not true if $\overline{I} \notin \Im$ for some $I \in \Im$.This may be seen from the following example.

Example 4.6 Let Q be a set of all rational numbers and let $\Im = \{\phi\}$.Take $X = Q \cup \{\phi\}$. Then $\overline{X}$ is strongly $\Im$- connected and hence X is $\Im$ - well linked but X is not strongly $\Im$ - connected.

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