MATHEMATICAL MODELING OF BRAZIL'S POPULATION GROWTH

[1]N.Sindhupriya , [2]Dr.T.Siva Subramania Raja , [3]K.sathya ,
[1]scholar , [2]Assistant professor , [3] Assistant professor
[1] Department of Mathematics 8-5-3G,Georgepuram,Nilakottai-624208, Dindigal,Tamilnadu.India.
[2] PG and research Department of Mathematics,National college,Trichy-001,Tamilnadu,India.
[3] Department of Mathematics,Amman college of Arts and science,Dindigal-02,

Abstract : The paper presents an approach for the variance of population growth in Brazil. Population of Brazil has been predicted with the help of an ordinary differential equation model known as logistic population model which is parameterized by growth rate along with carrying capacity human Population of Brazil . Secondary data for Brazil Populations (2006-2016) from international database is collected and the model proposed in this paper to find the birth ,death and population growth rate is found to be a very good fit with the actual data. So the logistic model is implemented to predict the future population growth rate for Brazil up to 2100 and the results are analyzed by using MATLAB software.

Keywords - Logistic growth model, Carrying capacity, vital coefficients, Annual growth rate.

I . INTRODUCTION

The "population growth rate" is the rate at which the number of individuals in a population increases in a given time period, expressed as a fraction of the initial population. Specifically, population growth rate refers to the change in population over a unit time period, often expressed as a percentage of the number of individuals in the population at the beginning of that period. A positive growth rate indicates that the population is increasing, while a negative growth rate indicates that the population is decreasing . Every government and collective sectors always require accurate idea about the future size of various entities like population, resources, demands and consumptions for their planning activities. Population sizes and growth in a country directly influence the situation of economy, policy, culture, education and environment of that country and determine exploring and cost of natural resources So, the study on population growth rate of any country has becomes very essential. Mathematical modeling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in life sciences. A mathematical model including dynamical systems, statistical models and differential equations involves variety abstract structures. Brazil is the largest country in both south america and latin america. Brazil is world's fifth largest country by both area and popoulation and also the largest country to have portuguese as an official language. Brazil population is equivalent to 2.83% of the total world population. Brazil rank numer 5 in the list of countries by population. Brazil population density is 24.5 people per square kilometer as of november 2016. Density of population is calculated as permanently settled population The current population of Brazil is 208846892 as of 2018. So the increasing trend in population is great threat to the nation. The use of the logistic growth model is widely established in many fields of modeling and forecasting [1]. First order differential equations govern the growth of various species. At first glance it would seem impossible to model the growth of a species by a differential equation since the population of any species always changes by integer amounts. Hence the population of any species can never be a differentiable function of time. However if a given population is very large and it is suddenly increased by one, then the change is very small compared to the given population [2]. Thus we make the assumption that large populations change continuously and even differentiable with time. The projections of future population are normally based on present population. In this paper, we will determine the carrying capacity and the vital coefficients governing the population growth of Brazil. Further this paper gives an insight on how to determine the carrying capacity and the vital coefficients, governing population growth, by using the least square method. This paper is organized as follows In Section 2 Mathematical Model of the paper is described. Section3 and Section4 give analysis on population growth rates of Brazil and predicted birth ,death and population growth rates of Brazil respectively. Conclusion of the study is given in Section5.

II . MODEL DESCRIPTION

- Let $I(t)$ indicate the population of a given species at time t
- Let 'α' be the difference between its birth rate and death rate.

If this population is isolated, then $\frac{d}{dt}I(t)$, the rate of change of the population, equals $\alpha\, I(t)$ where 'α' is a constant that does not change with either time or population. The differential equation governing population growth in this case is

$$\frac{d}{dt}I(t) = \alpha\, I(t) \tag{1}$$

where, t represents the time period and α, referred to as the Malthusian factor, is the multiple that determines the growth rate. This mathematical model of population growth was proposed by an Englishman, Thomas R. Malthus [3] in 1798. Equation (1) is a non-homogeneous linear first order differential equation known as Malthusian law of population growth. $I(t)$ takes on only integral values and it is a discontinuous function of t. However, it may be approximated by a continuous and differentiable function as soon as the number of Individuals is large enough [4]. The solution of equation (1) is

$$I(t) = I_0\, e^{\alpha t} \tag{2}$$

Hence any species satisfying the Malthusian law of population growth grows exponentially with time. This model is often referred to as *The Exponential Law* and is widely regarded in the field of population ecology as the first principle of population Dynamics. At best, it can be described as an approximate physical law as it is generally acknowledged that nothing can grow at a constant rate indefinitely. As population increases in size, the environment's ability to support the population decreases. As the population increases per capita food availability decreases, waste products may accumulate and birth rates tend to decline while death rates tend to increase. Thus it seems reasonable to consider a mathematical model which explicitly incorporates the idea of carrying capacity (limiting value). A Belgian Mathematician Verhulst [5], showed that the population growth not only depends on the population size but also on how far this size is from its upper limit i.e. its carrying capacity (maximum supportable population). He modified Malthus's Model to make the population size proportional to both the previous population and a new term

$$\frac{\alpha - \beta I(t)}{\alpha} \tag{3}$$

Where α and β are called the vital coefficients of the population. This term Reflects how far the population is from its maximum limit. However, as the population value grows and gets closer to $\frac{\alpha}{\beta}$ this new term will become very small and tend to zero, providing the right feedback to limit the population growth. Thus the second term models the competition for available resources, which tends to limit the population growth. So the modified equation using this new term is:

$$\frac{d}{dt}I(t) = \frac{\alpha I(t)(\alpha - \beta I(t))}{\alpha} \tag{4}$$

This is a nonlinear differential equation unlike equation (1) in the sense that one cannot simply multiply the previous population by a factor. In this case the population $I(t)$ on the right of equation (4) is being multiplied by itself. This equation is known as the logistic law of population growth. Putting $I = I_0$ for $t = 0$, where I_0 represents the population at some specified time, $t = 0$ equation (4) becomes

$$\frac{d}{dt}I = \alpha I - \beta I^2 \tag{5}$$

Separating the variables in equation (5)

$$\frac{1}{\alpha}\left(\frac{1}{I} + \frac{\beta}{\alpha - \beta I}\right) dI = dt \tag{6}$$

Integrating equation (6), we obtain $\int \frac{1}{\alpha}\left(\frac{1}{I}+\frac{\beta}{\alpha-\beta I}\right) dI = t + c$, so that

$$\frac{1}{\alpha}(\log I - \log(\alpha - \beta I)) = t + c \tag{7}$$

Using $t = 0$ and $I = I_0$ we see that $c = \frac{1}{\alpha}(\log I_0 - \log(\alpha-\beta I_0))$. Equation (7) become $\frac{1}{\alpha}(\log I - \log(\alpha-\beta I)) = t + \frac{1}{\alpha}(\log I_0 - \log(\alpha - \beta I_0))$. Solving for I yields

$$I = \frac{\frac{\alpha}{\beta}}{1+\left(\frac{\frac{\alpha}{\beta}}{I_0}-1\right)e^{-\alpha t}} \tag{8}$$

If we take the limit of equation (8) as $t \to \infty$, we get (since $\alpha > 0$)

$$I_{max} = \lim_{t\to\infty} I = \frac{\alpha}{\beta} \tag{9}$$

Next, we determine the values of α, β and I max by using the least square method. Differentiating equation (8), twice with respect to t gives

$$\frac{d^2 I}{dt^2} = \frac{C\alpha^3 e^{\alpha t}(C - e^{\alpha t})}{\beta(C+e^{\alpha t})^3} \tag{10}$$

Where $C = \frac{\frac{\alpha}{\beta}}{I_0} - 1$

At the point of inflection this second derivative of I must be equal to zero. This will be so, when

$$C = e^{\alpha t} \tag{11}$$

Let the time when the point of inflexion occurs be $t = t_k = y$. Then $C = e^{\alpha t}$ becomes $C = e^{\alpha t_k}$. Using this new value of C and replacing $\frac{\alpha}{\beta}$ by K equation (8) becomes

$$I = \frac{K}{1+e^{-\alpha(t-t_k)}} \tag{12}$$

Let the coordinates of the actual population values be (t, i) and the coordinates of the predicted population values with the same abscissa on the fitted curve be (t, I). Then the error in this case is given by $(I - i)$. Since some of the actual population data points lie below the curve of predicted values while others lie above it, we square $(I - i)$ to ensure that the error is positive. Thus, the total squared error, e, in fitting the curve is given by

$$e = \sum_{j=1}^{n}\left(I_j - i_j\right)^2 \tag{13}$$

Equation (13) contains three parameters K, α and t_k. To eliminate K we let

$$I = Kh \tag{14}$$

Where

$$h = \frac{1}{1+e^{-\alpha(t-t_k)}} \tag{15}$$

Using the value of I in equation (14) and algebraic properties of inner product equation (13), We have

$$e = \sum_{j=1}^{n}\left(I_j - i_j\right)^2$$

$= (I_1-i_1)^2 + (I_2-i_2)^2 + \ldots\ldots\ldots + (I_n-i_n)^2$

$= (Kh_1 - i_1)^2 + \ldots\ldots\ldots\ldots + (Kh_n - i_n)^2$

$= |(Kh_1 - i_1 \ \ldots\ldots\ldots \ Kh_n - i_n)|^2$

$= |(Kh_1 \ldots\ldots\ldots Kh_n) - (i_1, \ldots\ldots\ldots, i_n)|^2$

$= |KH - W|^2$

$= < KH - W, \ KH - W >$

$= K^2 < H, H > -2K < H, W > + < W, W >$

Where, $H = (h_1, h_2 \ldots\ldots, h_n)$ and $W = (i_1, i_2 \ldots.., i_n)$. Thus

$$e = K^2 < H, H > -2K < H, W > + < W, W > \qquad (16)$$

Taking partial derivative of equation (16) with respect to K and equating it to zero, we obtain $2K < H, H > -2 < H, W >= 0$. This gives

$$K = \frac{< H,W >}{< H,H >} \qquad (17)$$

Substituting this value of K into equation (16), we get

$$e = < W, W > - \frac{< H,W >^2}{< H,H >} \qquad (18)$$

This equation is converted into an error function substitute the approximate values of α and y *in* equation (17).Then find the value of K by using MATLAB program.

III . ANALYSIS OF GROWTH RATE OF BRAZIL

3.1. ANNUAL PERCENTAGE GROWTH RATE

$$Pop_{Future} = Pop_{Present} \times (1 + i)^n$$

Where Pop_{Future} = Future Population , $Pop_{Present}$ = Present Population,

i = Growth Rate (unknown) , n = Number of Years

Pop_{Future} =**188131059** (actually 2006) , $Pop_{Present}$ = **186020004** (actually 2005)

i = APG , n = 1

Example

In 2005 the population in Brazil was **186020004**. This grew to **188131059** in 2006. Then we can calculate annual percentage growth rate

N = 2008 – 2007 = 1

188131059= **186020004** $\times (1 + i)^1$

$$\frac{188131059}{186020004} = (1 + i)^1$$

1.011348538 = (1 + i)

APG = (1.011348538 - 1) $\times$ 100 = 1.13

3.2. PREDICTED GROWTH RATE OF BRAZIL IN 2100

Brith, Death and Actual Population values of Brazil, collected from International Data Base is given in Table 1

Table 1 : Brith, Death and Actual Population values of Brazil

s.no	years	**Actual population**	**Growth rate**	**Deaths**	**Growth rate**	**Births**	**Growth rate**
1	2006	188131059	1.13	1204039	1.13	3331801	-1.86
2	2007	190167417	1.08	1217071	1.08	3267076	-1.94
3	2008	192130270	1.03	1229634	1.03	3202812	-1.97
4	2009	194019058	0.98	1243662	1.14	3139228	-1.98
5	2010	195834188	0.93	1257255	1.09	3078513	-1.93
6	2011	197595498	0.89	1272515	1.21	3056802	-0.71
7	2012	199321413	0.87	1289610	1.34	3033672	-0.75
8	2013	201009622	0.84	1308573	1.47	3009114	-0.81
9	2014	202656788	0.81	1325375	1.28	2983108	-0.86
10	2015	204259812	0.79	1344030	1.40	2953597	-0.99
11	2016	205823665	0.76	1364611	1.53	2939162	-0.49

R = Average of APG = 1.03

$$\alpha = \frac{APG}{100} = \frac{1.13}{100} = 0.0113 = r$$

The Population growth rate of Brazil is approximately 1.13% per annum .We find that the values α and t_k (or y) are 0.0113 and 2100 respectively.

Using the MATLAB program ,We get the values of carring capacity ,vital coefficient value, error function, predicted values,plot the graph of predicted values with extended time.

Substituting the values of α and t_k in MATLAB program using equation (17) we get

$$I_{max} = K = 736783054 \quad (19)$$

This is the predicted carrying capacity or limiting value of the population of Brazil. Using equation (9) , we find that

$$\beta = \frac{0.0113}{736783054} = 1.533369434 \times 10^{-11} \quad (20)$$

This is another vital coefficient of the population. The error function

$$e = 2.6029924177 \times 10^{12}$$

Let t = 0 correspond to the base year 2006, then the initial population will be denoted by I_0. Where $I_0 =$ x =188131059

- [t=0(2006),t=1(2007),………………t=94(2100)]
- [2006 is base year and 2100 is end year]

Substituting the values of I_0 , $\frac{\alpha}{\beta}$ (or K) and α in the equation (8), we obtain

$$I = \frac{736783054}{1+\left(\left(\frac{736783054}{188131059}\right)-1\right)*(0.988)^t} \quad (21)$$

The equation (21) is used to compute the predicted values of the population.

t = end year – base year

$$t = 2100 - 2006 = 94 \quad (22)$$

Substituting the values t into the equation (21), we get

$$\frac{\alpha}{2\beta} = 368391527 \quad (23)$$

Thus, the population of Brazil is predicted to be 368391527 in the year 2100. This predicted population value is half of its carrying capacity.

Similarly predicted population values from 2006 to 2016 is calculated having 2006 as base year and is given in the following Table 2

Table 2: Actual population and Predicted population values

s.no	Years	**Actual population**	**Predicted population**
1	2006	188131059	188131059
2	2007	190167417	189827338
3	2008	192130270	191533572
4	2009	194019058	193249724
5	2010	195834188	194975754
6	2011	197595498	196711622
7	2012	199321413	198457282
8	2013	201009622	200212692
9	2014	202656788	201977802
10	2015	204259812	203752565
11	2016	205823665	205536930

The graph of actual and predicted population values against time is given in fig1

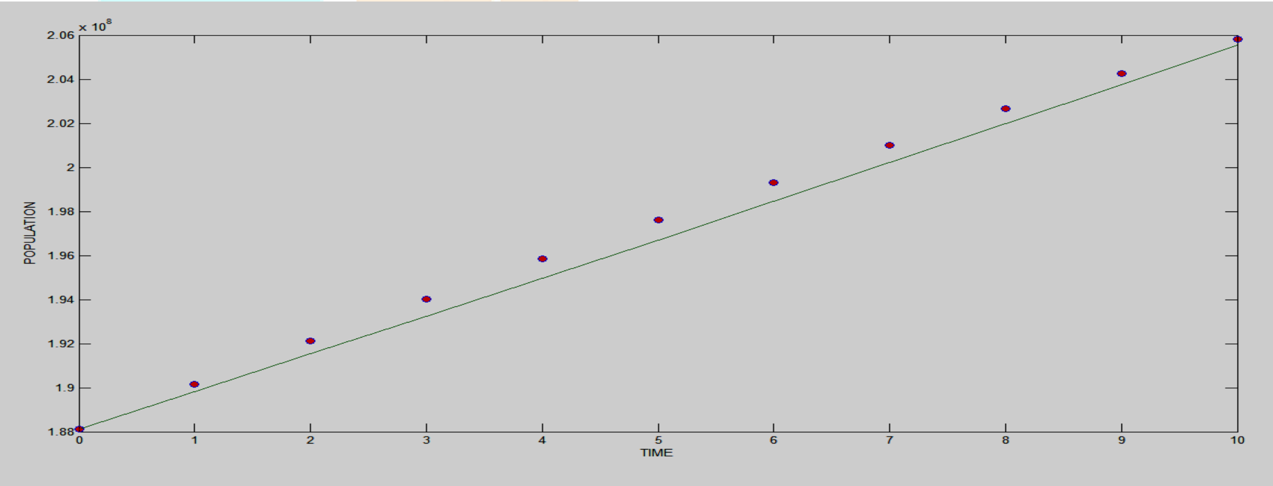

Figure 1

The graph of predicted population values against time is given in fig 2 .The values are computed using equation (21)

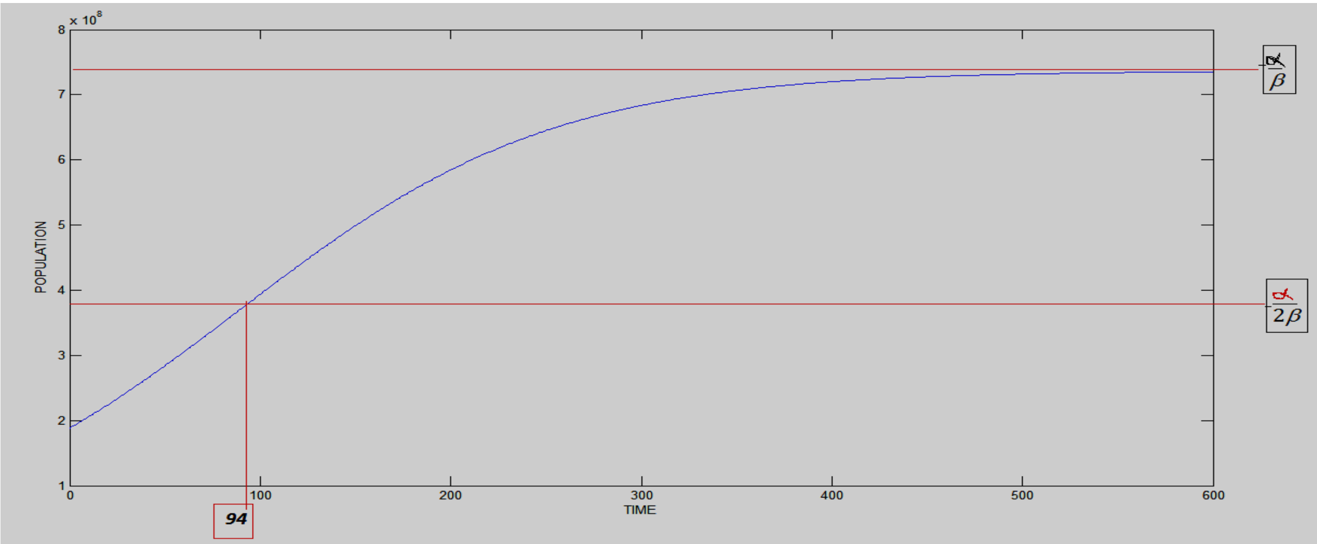

Figure 2

From fig 2 ,we understand that the population of Brazil will reach the carrying capacity after 600 years**.**

3.3. PREDICTED BIRTH AND DEATH RATE OF BRAZIL

The birth and death rate of Brazil is approximately -1.98% and 1.53% per annum .The predicted birth and death values from 2006-2016 is calculated using in MATLAB program (APPENDIX A) and is given in the following table 3

Table 3 : predicted birth and death values

s.no	Years	Actual- deaths	Predicted -death	Actual-births	Predicted- birth
1	2006	1204039	1204039	3331801	3331801
2	2007	1217071	1219803	3267076	3297531
3	2008	1229634	1235723	3202812	3263294
4	2009	1243662	1251799	3139228	3229098
5	2010	1257255	1268032	3078513	3194948
6	2011	1272515	1284421	3056802	3160851
7	2012	1289610	1300967	3033672	3126814
8	2013	1308573	1317669	3009114	3092843
9	2014	1325375	1334529	2983108	3058945
10	2015	1344030	1351545	2953597	3025126
11	2016	1364611	1368718	2939162	2991393

The graph of actual and predicted Death values against time is given in fig 3

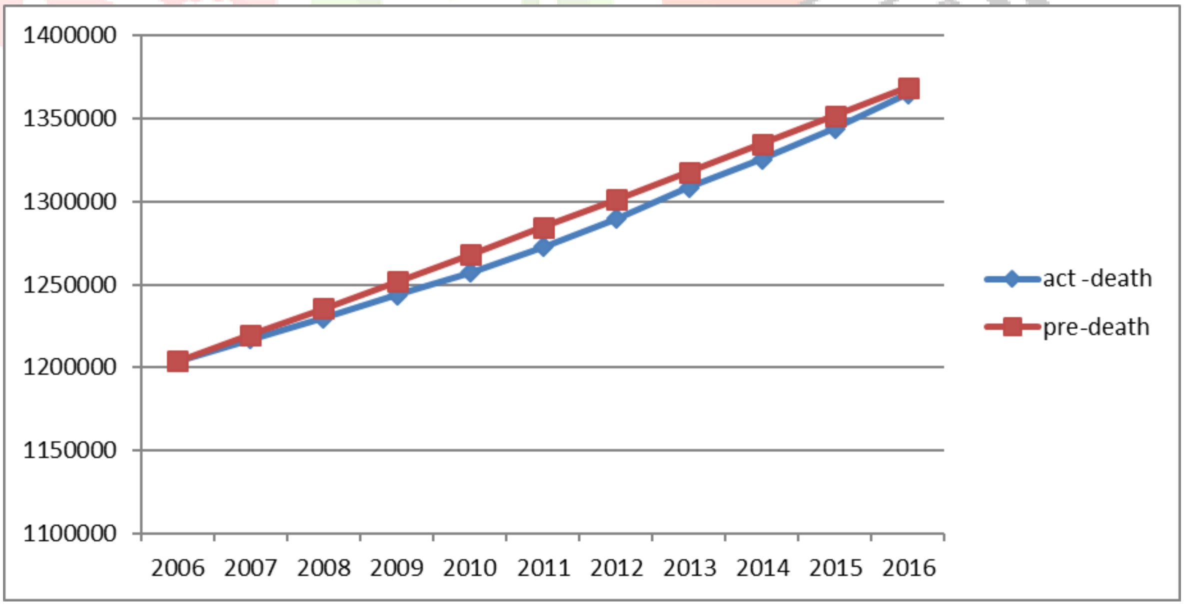

Figure 3

The graph of actual and predicted Birth values against time is given in fig 4

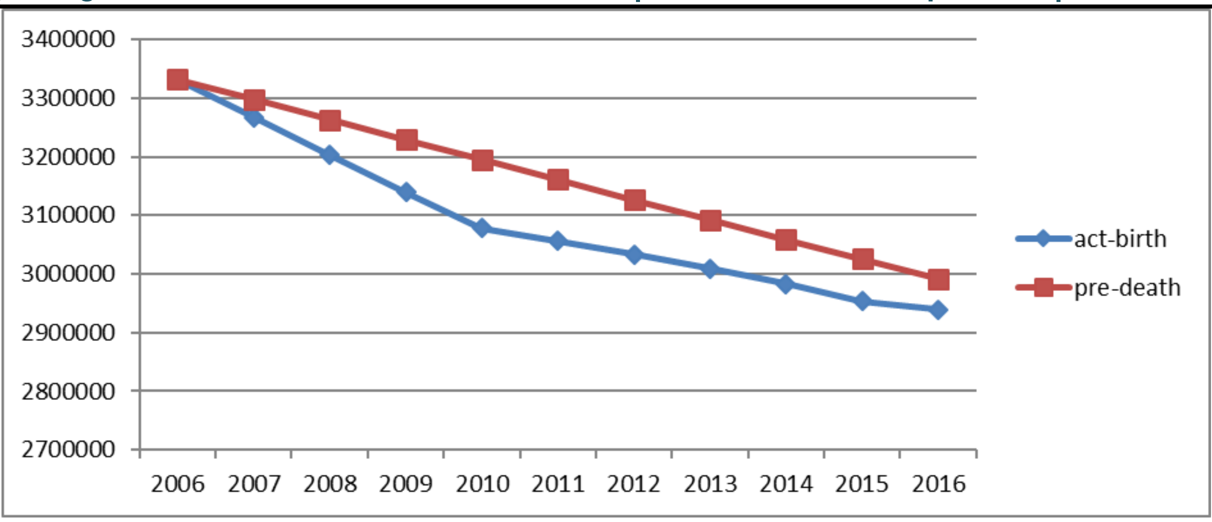

Figure 4

IV. RESULTS AND DISCUSSION

4.1 ANALYSIS OF GROWTH RATE OF BRAZIL FROM 2019 - 2050.

Table 4 : Shows the projected birth , death and population of Brazil

s.no	*Years*	*future population of Brazil*	*Births*	*Deaths*	s. no	*Years*	*future population of Brazil*	*Briths*	*Deaths*
1	2019	210947091	2890766	1421175	17	2035	241181803	2374513	1724384
2	2020	212769312	2857436	1438973	18	2036	243144432	2343686	1744611
3	2021	214600849	2824222	1456927	19	2037	245115134	2313057	1764982
4	2022	216441640	2791130	1475037	20	2038	247093813	2282629	1785498
5	2023	218291620	2758166	1493302	21	2039	249080373	2252407	1806157
6	2024	220150722	2725334	1511721	22	2040	251074716	2222393	1826957
7	2025	222018878	2692642	1530295	23	2041	253076740	2192592	1847897
8	2026	223896017	2660094	1549022	24	2042	255086345	2163007	1868976
9	2027	225782065	2627696	1567903	25	2043	257103425	2133642	1890192
10	2028	227676948	2595453	1586937	26	2044	259127875	2104500	1911545
11	2029	229580589	2563370	1606122	27	2045	261159586	2075584	1933032
12	2030	231492910	2531452	1625459	28	2046	263198450	2046897	1954652
13	2031	233413828	2499705	1644946	29	2047	265244355	2018442	1976404
14	2032	235343263	2468133	1664583	30	2048	267297186	1990221	1998285
15	2033	237281128	2436740	1684369	31	2049	269356830	1962237	2020294
16	2034	239227338	2405532	1704303	32	2050	271423169	1934493	2042430

V . CONCLUSION

In conclusion we found that the predicted carrying capacity for the population of Brazil is 736783054. Population growth of any country depends on vital coefficients also. In this case of Brazil we found out that the vital coefficients α *and* β are 1.13% and 1.533369434 $\times 10^{-11}$ respectively. According to this model birth rate is -1.98% ,death rate is1.53% and population growth rate is 1.13% per annum in Brazil . From logistic model we also found out that the population of Brazil is expected to grow rapidly when there are 368391527 (half of the carrying capacity) populations in the year 2100. The average annual percent change in the population,resuling from a surplus(or deficit) of births over deaths and the balance of migrants entering and leaving a country. The rate may be positive or negative.the growth rate is a factor in determining how great a burden would be imposed on a country by the changing needs of its people for infrastructure ,resources and jobs. Rapid population growth can be seen as threatening by neighboring countries.

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APPENDIX(A):

y=2000;
r=-0.0198;
x=3331801;
t=[2006 to2016];
i=[3331801 to 2953567];
h=1./(1+exp(-r*(t-y)))
h=h'
i=i'
t=t'
e=(i'*i)-((h'*i)^2/(h'*h))
K=(h'*i)/(h'*h)
P=K./2
t=0:44;
format long
I=K./(1+((K/x)-1)*(1.02).^t)
	y=2100;
	r=0.0153;
	x=1204039;
	t=[2006 to 2016];
	i=[1204039 to 1364611];
	h=1./(1+exp(-r*(t-y)))
	h=h'
	i=i'
	t=t'
	e=(i'*i)-((h'*i)^2/(h'*h))
	K=(h'*i)/(h'*h)
	P=K./2
	t=0:44;
	format long
	I=K./(1+((K/x)-1)*(0.984).^t)