# SELF-COMPLEMENTARY GRAPH

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***Absrtact:*** In this paper a new algorithm is given for the construction of self-complementary graphs, and then concerning self-complementarity , isomorphism, and generating the self-complementary graphs and also presented the self-converse graphs are presented..

### *Introduction*

Graph theory is the study of graphs .which are mathematical structures used to model pairwise relations between objects .A graph in this context is made up of vertices ,nodes or points which are connected by edges arcs or lines. A graph may be undirected meaning that there is no distinction between the two vertices associated with each edge or its edges may be directed from one vertex is another.

A self-complemantary graph is a graph which is isomorphic to its complement .this simplest non-trivial self-complementary graph are the 4-vertex path graph these graphs have a high degree of structures .In this paper a new algorithm for consisting self-complementary graphs and resuts concerning structural properties and adjacency matrix and eigen value properties of these matrices are presented. suffice to say that the problem of recognizing self-complementary graphs ,and the problems of checking two SC-graphs for isomorphism problem.

Self-complementary graphs studied in the four decades since the seminal papers of sachs ( Uber selbstkomplementare graphen, Publ Math Drecen 9 (1962) 270-288 MR 27-1934), Ringel ( selbstkomplementare graphen, Arch. Math. 14 (1963) 354-358. MR 25:22) and read (on the number of self-complmentary graph and digraph, J. Lond,Math,Soc.38(1963) 99-104. MR 26:4339). The areas convered include connectivity,degree sequence,self-complementarity,isomorphism and generating self-complementary graphs and also self- converse graphs.

### *Definition 1.1*

The degree of an edge is the ordered pair (d(u),d(v)) of degrees of its vertices and the edge degree sequence is the sequence of degree of its edge .An edge degree sequence with at least one self -graph realization is called a potentially edge self-complementary .If all its realizations are self-complementary it forcibly edge self-complement.

### *Definition 1.2*

A complement or inverse of a graph G is a graqph H on the same vertices such that two distinct vertices of H are not adjacent if and only if they are no adjacent in G .That is,to generate the complement of a graph one fills in all the missing edges required to form a complete graph , and removes all the edges that were previously there .It is not ,however , the said complement of the graph; only the edges are complemented

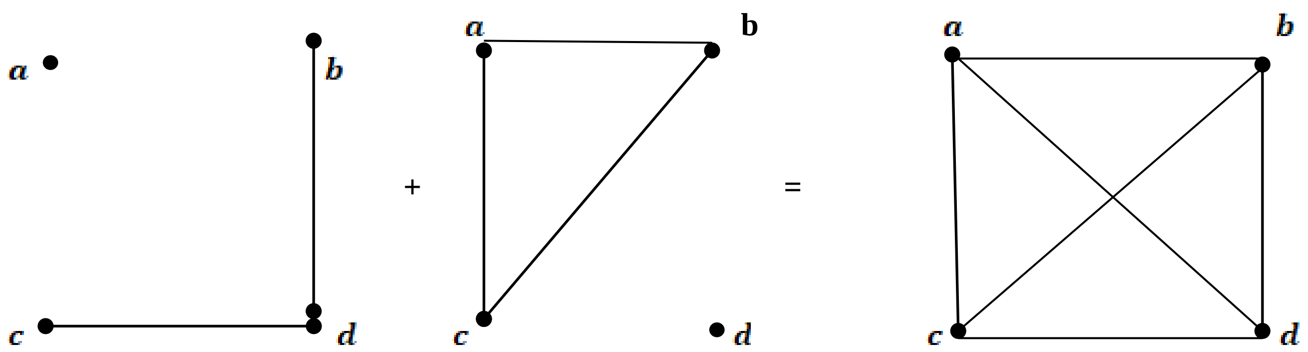
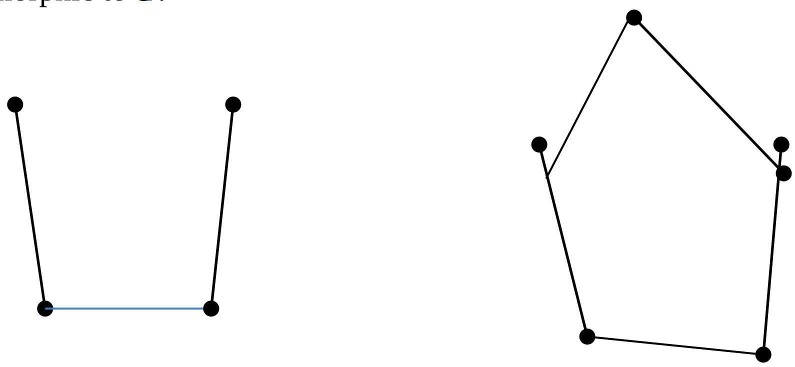



Figure 1.1

***Definition 1.3***

Let G = (V,X) be a graph. The G of $\bar{G}$ is defined to be the graph which has V as its set of points and two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G. G is said to be self-complementary graph if G is isomorphic to $\bar{G}$.



4- vertex path graph 5-vertex cycle graph

Figure 1.2

***2. The construction Algorithm and Structural Properties***

***Observation 2.1***

If G is a self-complementary graph with n vertices ,then n $\equiv$ 0,1(mod 4). This follows because G must have half of the possible $(n|2\ )$ edges .Hence ,n(n-1) /4 must be an integer.

***Observation 2.2***

Since the complement of any disconnected graph is connected ,any self-complementary graph is connected.

***Result 2.3***

If G is self-complementary with n vertices and $\tau\ (G)$=$\bar{G}$ then if n$\equiv$ 0(mod 4) each cycle of has length divisible by 4 and if n$\equiv$0(mod 4) $\tau$ has exactly one cycle of length 1 and all other cycles have length divisible by 4.

***Constuction Algorithm 2.4***

Assume without loss of generality ,that the symbols in are numbered consecutively from 1 to n ,and that the cycles are non-decreasing lengths $4_{k1}$ ,$4_{k2}$,…(expect for a possible 1-cycle (n)at the end).Here $k_{1,}k_2$ …are powers of 2.

The symbols will constitute the range of symbols 2,3….2$k_1$+1 of the first cycle ,the first 4$k_1$ symbols of each other cycle and the symbol n if (n) is a 1- cycle consrtruction of graph G with the vertices labeled 1,2…n and therefore identify the symbols in $\tau$ with the vertices in G .

Consider,

The pairs of symbols (1,j) where or not 1 and j will be adjacentin in G .Once these choices have been made the same choices must apply to the pairs.($\tau^{2i}$(1),$\tau^{2i}$(j)),for i=1,2….$2k_{ij}$,Where j is in a cycle of length $4k_{ij.}$ (if j= n =4k+1 then let i= 1,2,…..$2k_1$)The opposite adjacency relations must apply to the pairs.($\tau^{2i-1}$ (1),$\tau^{2i-1}$(j)) where, i is above.this completes the first stage of the algorithm we next reduce the problem by replacing the permutation $\tau$ by the simpler permutation $\tau^*$ ,on the n-$4k_1$ symbols, obtained from $\tau$ by deleting its first cycle.

Now apply to $\tau^*$ the procedure described .then delete the first cycle of $\tau^*$and continue this way until no cycles remain.The algorithm wil terminate since $\tau$ has a finite of symbols.

***Theorem 2.5***

If a self –complementary graph having K vertices posseses a complementing permutation consisting of cycles of equal length ,then .

(i) A(G) is symmetric 2k$\times$ 2k matrix $A^1$ whose entries are polynomials in the matrix S $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The diagonal entries are s or –s and all other entries are $\pm$e $\pm$ s where e is the 2$\times$ 2 identity matrix.

(ii) If we form from $A_1$ the 2k $\times$ 2k matrix ,$B_1$,by setting e = s = 1,and $B_2$ by setting e = -s = 1 then the eigen values of A(G) are precisely those of $B_1$ together with those of $B_2$.

***proof***

Let $\tau$(G) = $\bar{G}$ where $\tau$ = (1,2,…4$K_1$ ) (4$k_1$+1….8$k_1$) ….( …..4k) form $A_1$ from A(G) by labeling its rows and columns in the orer .

1,2$k_1$+1,2,2$k_1$+2,…..2$k_1$,4$k_1$

4$k_1$ + 1,6$k_1$+1,4$k_1$+1,,,6$k_1$+2…6$k_1$,8$k_1$.

……………………..

4(k-$k_1$ )+1,4 (k-$k_1$ ) + 2$k_1$+1,…4k_ 2$k_1$,4k.

For the entry at row i and column j (according to this new ordering ) a(i ,j) consider the consecutive pairs .

1,(2 $k_1$+1) ,2 (2$k_1$+2 ) ….,(4k -2$k_1$,4 k) and the 2 x 2 submatrix of $A_1$ formed by two of these pairs .

$$\begin{array}{c} \\ \vdots \\ t \\ 2k_1+t \\ \vdots \end{array} \begin{array}{c} \dots(m\ ,2k_1+m\ )\dots \\ \begin{bmatrix} \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ \\ \end{bmatrix} \end{array}$$

There are two cases:

***Case i*** If t = 0

Then, a block on the diagonal of $A_1$ and so a = d = 0

Now, b = a ( t ,2$k_1$+1 ) and

c = a (2$k_1$ +t ,m ) = a($\tau^{2k1}$ (t), $\tau^{2k1}$ (2$k_1$+ m )

= $(-1)^{2k1}$ a (t,2$k_1$ +m) = a(t,2$k_1$ +m)

= b

Therefore, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$= $\pm$ s

***Case ii*** If t $\neq$ m

Then, a = a (t,m) = $(-1)^{2k1}$ a (t,m) =a ($\tau^{2k1}$ (t),$\tau^{2k1}$ (m) )

=a ($2_{k1}$+t , $2_{k1}$+m = d

and

b = a(t ,$2k_1$+m) =$(-1)^{2k1}$ a(t,$2k_1$+m)

Therefore,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm e \pm s$$

This completes the proof of (1)

(Note that ,in both of the above cases ,the proof relied on the fact that all cycles of $\tau$ have length $4k_1$.)

(ii) The matrices e and s are simultaneously diagonalized by the matrix r = $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

In fact, $r^{-1}(\pm s)\, r = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$

Also note that

$$r^{-1}(\pm e \pm s) = \begin{bmatrix} \pm 2 & 0 \\ 0 & 0 \end{bmatrix}$$

and that

$$r^{-1}(\pm e \pm s) = \begin{bmatrix} 0 & 0 \\ 0 & \pm 2 \end{bmatrix}$$

*Let,*

$$R = \begin{bmatrix} r & & 0 \\ & r & \\ 0 & & r \end{bmatrix}$$

have 2k diagonal blocks equal to r. Then, A =$R^{-1}$ $A_1$R

Will be composed of 2 x2 blocks of the form listed in (5) and (6) it is readily seen that the i-j entry of $A_2$ is zero. Whenever, i +j ≡ 1 (mod 2 )

Therefore,

If transform $A_2$ by the 4k x4k permutation matrix P whose columns are respectively columns.

1,3,5…4k -1,2,4,6….4k of the unit matrix of dimension 4k

$p^{-1}A_2 p = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ Where C,D and 0 are matrices p.

The entries of C are the upper-left entries of the 2 x 2 blocks in $A_2$ and the entries of D are the lower –right entries of these 2 x 2 blocks .

Now observe that in the 2 x 2 matrices in (5) and (6) let e = s =1 obtain the upper –left entries and let e = -s = 1 the lower –right entries .

Therefore C = $B_1$, D = $B_2$.

***Observation 2.6***

The set of automorphisms and complementing permutations of an s.c.graph G forms a group in which the automorphisms group of G is a subgroup of index 2. If follows that G has a many complementing permutations as automorphisms.

***Observation 2.7***

The automorphisms group of an self-complementary graph G is a non-trivial graph.

***Theorem 2.8***

Let $\tau$ (G) = $\bar{G}$ Where $\tau$ = ( 1.,2.,…..4k); Then,

(i) Each odd-labeled vertetx of G is adjacent to exactly k even –labeled vertices and each even-labeled vertex is adjacent to exactly k odd-labeled vertices ,

(ii) G has vertices of two degrees :for some r such that k$\leq$ r $\leq$ 3k -1.there are 2k vertices of degree r and 2k vertices of degree 4k – 1-r .Moreover for every such r ,at least one self complementary graph G with $\tau$(G) = $\bar{G}$ exists having 2k vertices of degree r and 2k vertices of degree 4k-1-r.

***proof***

(i) Note that,

A(1,2i) = $(-1)^{4k+1-2i}$ a( $\tau^{4k+1-2i}$ (1),$\tau^{4k+1-2i}$)

= -a (4k = 2 -2i, 1)

= -a (1, 4k+2-2i).

Hence,

1 is adjacent to 2i if and only if it is not adjacent to 4k +2 -2i .therefore,1 is adjacent to exactly half 45r,of the 2k even-labeled vertices.

By the construction algorithm,

This implies that every odd-labeled vertex is adjacent to exactly k even-labeled vertices.

Similarly,

Any even-labeled vertex is adjacent to exactly k of the odd-labeled vertices.

## *3. The isomorphism problem*

***Definition 3.1***

One of the most basic tasks when dealing with a particular class of graph is to distinguish members of the class form each other, un form non-members. These are the "isomorphism problem" and the recognition problem, respectively, for that class.For graphs in general the isomorphism problem is as yet intractable-there is no known polynomial solution, and it is not even known if one exists.

Obviously,any test for determining wheather two graphs or digraphs are isomorphic will also work for SC-graphs and SC-digraphs. It can also be used to tell us wheather a given graphs is self-complementary, as we just need to check whether it is isomorphic to its complement.

***Theorem 3.2***

P=NP if and only if the recognition or isomorphism of SC-graphs is NP-complete.

***Proof***

If P = NP than the recognition or isomorphism or SC-graph must we polynomial, and thus NP complete.

Conversely,

If they are NP-complete, than graph isomorphism must also be NP-complete and thus, by kobler etal.,

We have, P=NP.

Dor and Tarsi settled a conjecture by holyer, to the effect that, for a given connected graph H with three edges or more, determining whether a graph G has a decomposition into factors isomorphic to H is NP-complete.

The problem of self-complementarity is some what related related, in this case the graph H can vary, but always have n(n-1)/4 edges, and G is always Kn,

Where n = |V(H)| .

***Theorem 3.3***

The digraph isomorphism problem is polynomial if and only if the self-complementary digraph isomorphism problem is polynomial.

***Proof***

Let us say we want to check weather two digraphs $D_1$and $D_2$ on n vertices are isomorphic. We form, $S_{D_1,D_2}$ by substituting $D_1$ and $D_2$ for the vertices of $P_2$ . Obviously, if $D_1$and $D_2$ are isomorphic.

Then,

$$D_1 \rightarrow \overline{D_2} \qquad \overline{D_1} \rightarrow D_2$$

Figure 3.1

$S_{D_1,D_2}$ is self-complementary. Conversely, let $S_{D_1,D_2}$ be self-complementary. Since every vertex in the copy of $D_1$ has out degree at least n,and every vertex in the copy of $\overline{D_2}$ has out degree at most n-1 ,any antimorphism $S_{D_1,D_2}$ of will map $D_1$to $\overline{D_2}$. So, $D_1 \cong D_2 \leftrightarrow S_{D_1,D_2}$ is self-complementary. If instead of $D_2$ we use the converse , $\overline{D_2}$ we see that , $D_1 \cong D_2 \leftrightarrow S_{D_1,D_2}$ is self-converse.

Moreover,

If we can check for self-complemenatrity or self-converseness in $O(n^r)$ time ,for some constant r, then we can check any two digraphs for isomorphism$O(2^r n^r + n^2) = O(n^r)$ time.

### *4. Generating self-complementary graphs*

***Theorem 4.1***

A permutation $\sigma$ is an antimorphism of some SC-graph if and only if

A) All the cycles of $\sigma$ have length a multiple of 4,or

B) $\sigma$ has one fixed vertex ,all other cycles having length a multiple of 4.

***proof***

Describe an algorithm to construct all SC-graphs which have $\sigma$ as an antimorphism,and show that this works only when condition A or B is satisfied .Let be $K_n$ the complete graph on n vertices .

I. Take an arbitrary pair of vertices {a,b} and colour the edge between them red. colour all the edges $\sigma^{2i}\{a,b\} = \{\sigma^{2i}(a); \sigma^{2i}(b)\}$ red ,and all the edges $\sigma^{2i-1}\{a,b\}$ blue,for each integer i.

II. Repeat step 1 for any uncoloured edges ,until all edge orbits have been coloured .

III. From each orbit ,choose either the red edges or the blue edges (we may choose red edges from one orbit and blue from another) the chosen edges then from a self-complementary graph.if there are S orbits ,then there are 2s different ways of making the choices in C,though some of them may give us isomorphic graphs ,and some SC-graphs may also be associated with other antimorphisms with different cycle lengths( such as $\sigma^3$if $\sigma^5$ order is not a power of 2). Evidently there are no other SC-graphs with antimorphism $\sigma$ ,apart from the Sins produced here.

The colouring described in A is well-defined ,unless some edge {a,b} is coloured both red and blue ,that is ,unless $\{a,b\} = \{\sigma^{2i}\{a,b\}$ for some i.

This can happen in three ways:

***Case 1***

a and b are in the same cycle ,and we have $a = \sigma^{2i+1}(a)$ and $b = \sigma^{2i+1}(b)$ . Then the cycle length divides $2i + 1$, in particular ,the cycle length is odd.conversly,if there were a cycle of odd length $2i + 1 > 1$ any two vertices of the cycle would give rise to this problem ,so $\sigma$ can only contain even-length cycles and fixed vertices.

***Case 2***

a and b are in the same cycle, and we have $a = \sigma^{2i+1}(b)$ and $b = \sigma^{2i+1}(a)$ .then the cycle has length $4i + 2$. Conversely, if there were a cycle $\{v_1, v_2, \dots \dots . . v_{4i+2}\}$. Then $v_1$ and $v_{4i+2}$ would give rise to this problem ,so cannot $\sigma$ contain any cycles of length 2 (mod 4).

***Case 3***

a and b are in different cycles .then we must have $a = \sigma^{2i+1}(a)$ and $b = \sigma^{2i+1}(b)$ this can only happen if both cycles have odd length .since, we have ruled these out, except for the fixed vertices we see that we cannot have two or more fixed vertices.so,the algorithm works if has at most one fixed vertex ,and all other cycles have length a multiple of 4.

## *5. Self-Converse digraphs*

The construction of self-converse digraphs is significantly different.the orbits are now defined by $\sigma^{2i}\{a, b\} = \{\sigma^{2i}(a); \sigma^{2i}(b)\}$ In particular,

* There is no problem if we have $a = \sigma^{2i+1}(a)$ and $b = \sigma^{2i+1}(b)$ for some;it just means that particular arc orbit will include both $(a, b)$and $(b, a)$ so now, any cycle lengths are admissible.

* A 2-cycle say $(a, b)$ will give us two separate arc orbits $(a, b)$and $(b, a)$ .

For each arc orbit we can choose either to include all the arcs of that orbit or none;of course we make each choice independently.If we make the same choice for each orbit ,we obtain the trivial self-converse digraphs kn vector and the null digraph.

If we want to construct self-converse r-multigraphs,then in each orbit we replace every arc by a bundle of t arcs ,for some $r < 1$,where the choice of t is made independently for each orbit.For $r = 1$ we obtain the usual self-converse digraphs as described in the previous paragraph.

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