A STUDY ON FOURIER SERIES AND FOURIER LAPLACE TRANSFORM

NAME: A.SILAMBARASI, GUIDE : MR.R VAITHYALINGAM
DEPARTMENT OF MATHEMATICS,
PRIST UNIVERSITY, TANJORE (PUDUCHERRY BRANCH) TAMIL NADU.

ABSTRACT:

This article is intended for first year undergraduate student. This paper presents the generalization of two dimensional Fourier-Laplace transform in the distributional sense.

KEY WORD:

Fourier series , , Laplace Transform, Fourier transform $L^2(R)$, Test functions and distribution

1.INTRODUCTION:

The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform (now called z transform) in his work on probability theory.

$$Z = \int X\ (x)\, e^{ax}\, dx \quad \text{and } Z = \int X\ (x)\, e^{A}\, dx$$

as solutions of differential equation but did not pursue the matter very far.

$$\int X(x) e^{-ax} a^x\, dx,$$

These types of integrals seem first to have attracted Laplace's attention in 1782 where he was following in the spirit of Euler in using the integrals themselves as solutions of equations.

However,1785, Laplace took the critical step forward when, rather than just looking for a solution in the form of an integral, he started to apply the transforms in the sense the was later to become popular. He used an integral of the form:

$$\int x^s\, \Phi(x) dx,$$

Laplace also recognized that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply to a limited region of space as the solution were periodic.

2.TRIGONOMETRIC SERIES AND FOURIES SERIES:

Definition 2.1.

A series of the form

$$c_0 + \sum_{n=1}^{\infty} (a_n \cos \text{nx} + b_n \sin \text{nx})$$

is called a TRIGONOMETRIC SERIES, where c_0, a_n, b_n are real numbers.

Definition 2.2.

Let f $\in L^1$ [$-\pi$, π]The FOURIES SERIES of f is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

The series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad with \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx$$

is also called the FOURIER SERIES of ' f '. The coefficients c_n are called the FOURIER COEFFICIENT and are usually denoted by $\hat{f}$ (n) , i.e.,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-inx} \, dx, n \in Z$$

the sum

$$S_N(f, x) = \sum_{n=-N}^{N} (\hat{f}(\text{n}) e^{inx})$$

is called the N-th partial sum of the Fourier series.

3. FOURIER-PLANCHERAL TRANSFORM ON $L^2(R)$ AND BANACH ALGEBRA

The Banach Algebra $L^1(R)$

Now, we define a multiplicative structure on $L^1(R)$

Definition 3.1.1

For f, g in $L^1(R^d)$, we define the convolution of f and g , denoted by $f * g$, is defined by

$$(f * g)(x) = \int_{R^d} f(x - y) g(y) \, dy,$$

For almost all $x \in R^d$.

APPROXIMATE IDENTITY

Definition 3.2.1

By an approximate identity we mean a family $\{e_\lambda : \lambda > 0\}$ in $L^1(R^d)$ such that

$$\|f * e_\lambda - f\|_1 \to 0 \ as \ \lambda \to 0.$$

Now, we specify an approximate identity for $L^1(R)$The same can be extended to the case of $L^1(R)$ for appropriate changes.

Here onwards, we consider a non-negative measurable function ψ which satisfies

$$\int_R \psi(x)dx = 1$$

and let

$$\psi_\lambda(x) = \frac{1}{\lambda}\psi\left(\frac{x}{\lambda}\right), \ x \in R, \lambda > 0$$

Then we have

$$(f * \psi_\lambda)(x) = \int_R f(x-y)\psi_\lambda(y)dy$$

$$= \int_R f(x-\lambda_s)\psi_\lambda(y)\,ds$$

Indeed

$$(f * \psi_\lambda)(x) = \int_R f(x-y)\psi_\lambda(y)dy$$

$$= \int_R f(y)\psi_\lambda(x-y)dy$$

$$= \int_R f(y)\frac{1}{\lambda}\psi\left(\frac{x-y}{\lambda}\right)dy$$

$$= \int_R f(x-\lambda s)\psi(s)ds$$

FOURIER-PLANCHERAL TRANSFORM ON $L^2(R)$

Definition 3.3.1

Fourier transform of functions in $L^2(R)$ by extending the known definition from $L^1(R) \cap L^2(R)$ to all of $L^2(R)$.

Lemma 3.3.2

Let $f \in C_b(R)$. Then

$$\lim_{\lambda \to 0}(f * \psi_\lambda)(x) = f(x) \quad for\ every\ x \in R$$

where ψ_λ is as in set $\{\psi_\lambda : \lambda > 0\}$ is an approximate identity for $L^1(R)$.

Proof:

Note that for every $x \in R$,

$$(f * \psi_\lambda)(x) - f(x) = \int_R f(x-\lambda s) - f(x)\psi(s)\,ds.$$

Note that

$$|f(x-\lambda s) - f(x)|\psi(s) \leq 2\|f\|_\infty\psi(s) \ with\ \psi \in L^1(R)\ and$$

$$[f(x-\lambda s - f(x)]\psi(s) \to 0 \ as\ \lambda \to 0$$

As f is continuous at x

4. TEST FUNCTIONS AND DISTRIBUTIONS

Test functions and distribution

Let Ω be an open subset of R^d, We shall denote the vector space $C_c^\infty(\Omega)$ by D(Ω) and call this space as **Space of test functions.**

Definition 4.1.1.

A sequence φ_n) in D(Ω) is said to converge to ∈ D(Ω) if

(i) There exists a compact set K ⊂ Ω such that φ_n) ⊆ K for all $n \in N$ and
(ii) $\partial^\alpha \varphi_n \to \partial^\alpha$ φ uniformly on Ω for every $\alpha \in N_0^d$

Definition 4.1.2.

A distribution on Ω is a linear function u on D(Ω) such that for every φ_n) in D(Ω), $\varphi_n \to \varphi$ in D(Ω) implies $u(\varphi_n) \to u(\varphi)$.

The set of all distribution on Ω is denoted by D′ (Ω).

Definition 4.1.3.

A sequence u_n) of distributions on Ω is said to converge to a distribution u on Ω if $u(\varphi_n) \to u(\varphi)$ for every φ D(Ω).

Definition 4.1.4.

A distribution u on Ω is called a regular distributions if $u = u_f$ for some $f \in L^1_{loc}(\Omega)$ and in that case u_f is said to be the distribution 1 generated by f.

Theorem 4.1.5.

There exists a sequence u_n of regular distributions which converge to a delta distribution. In fact, taking $f_n := \frac{n}{2} \chi E_n$,

Where

$E_n := \{x \in \Omega : |x - a| < 1/n\}$,

$u_{f_n} \to \delta_u \; as \; n \to \infty$.

Proof:

Let $f_n := \frac{n}{2} \chi E_n$, where

$E_n := \{x \in \Omega : |x - a| < 1/n\}$,

and let $u_n := u_{f_n}$.

Let $\varphi \in$ D(Ω. Then

$$u_n(\varphi) = \frac{n}{2} \int_{|x-a|<1/n} \varphi(x)\, dx$$

Note that

$$u_n(\varphi) = \frac{n}{2} \int_{|x-a|<1/n} \varphi(x)\,dx$$

$$= \frac{n}{2} \int_{|x-a|<1/n} [\varphi(x) - \varphi(a)]dx + \varphi(a)$$

and

$$= \frac{n}{2} \int_{|x-a|<1/n} [\varphi(x) - \varphi(a)]dx$$

$$\leq \max_{|x-a|<1/n} |\varphi(x) - \varphi(a)| \to 0 \ \ as \ \ n \to \infty$$

Thus, $u_n(\varphi) \to \varphi(a) \ \ as \ \ n \to \infty.$

Definition 4.1.6.

For $\varepsilon > 0, an\ \varepsilon -$ impulse function is a non-negative function $\delta_\varepsilon(x)$ defined for $-\infty < x < \infty$ such that

1. $\int_{-\infty}^{\infty} \delta_\varepsilon(x)\,dx = 1,$

2. $\delta_\varepsilon(x) = 0 \ for \ |x| > \varepsilon,$

3. $\delta_\varepsilon(0) \to \infty \ as \ \varepsilon \to 0.$

Theorem 4.1.7.

For $\varepsilon > 0, if\ \delta_\varepsilon$ is an $\varepsilon -$ impulse function, then $u_{\delta_\varepsilon} \to \delta_0 \ as \ \varepsilon \to 0, where\ \delta_0$ is the delta- distribution at 0.

Proof:

The proof is along the sme line as that of Theorem (4.1.5)

Let φ be a continuous function defined on R and $\delta_\varepsilon(x)$ is an $\varepsilon -$ impulse function.

Then we have

$$\int_{-\infty}^{\infty} \varphi(x)\,\delta_\varepsilon(x)dx = \int_{-\epsilon}^{\epsilon} \varphi(x)\,\delta_\varepsilon(x)\,dx.$$

Hence,

$$\left| \int_{-\infty}^{\infty} \varphi(x)\,\delta_\varepsilon(x)dx - \varphi(0) \right| = \left| \int_{-\epsilon}^{\epsilon} \varphi(x)\,\delta_\varepsilon(x)\,dx - \int_{-\epsilon}^{\epsilon} \varphi(0)\,\delta_\varepsilon(x)\,dx \right|$$

Thus we have

$$\left| \int_{-\infty}^{\infty} \varphi(x)\,\delta_\varepsilon(x)dx - \varphi(0) \right| \leq \int_{-\epsilon}^{\epsilon} |\varphi(x) - \varphi(0)|\,\delta_\varepsilon(x)dx.$$

Since φ is continuous, for any given $\alpha > 0$, there is an $\varepsilon > 0$ such that $|\varphi(x) - \varphi(0)| < \alpha$ *whenever* $|x| < \varepsilon$.

Hence , for such an $\varepsilon > 0$,we have

$$\left| \int_{-\infty}^{\infty} \varphi(x)\, \delta_\varepsilon(x) dx - \varphi(0) \right| \leq \int_{-\varepsilon}^{\varepsilon} |\varphi(x) - \varphi(0)|\, \delta_\varepsilon(x) dx \leq \alpha \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x) dx = \alpha$$

That is for every $\alpha > 0$, there is an $\varepsilon > 0$ such that

$$\left| \int_{-\infty}^{\infty} \varphi(x)\, \delta_\varepsilon(x) dx - \varphi(0) \right| < \alpha.$$

Thus,

$$\int_{-\infty}^{\infty} \varphi(x)\, \delta_\varepsilon(x) dx - \varphi(0) \quad as \ \ \varepsilon \to 0.$$

and hence, $u_{\delta_\varepsilon}(\varphi) \to \delta_0(\varphi) \quad as \quad \varepsilon \to 0$. where δ_0 is the delta distribution at 0.

5 . CONCLUSIONS:

In this paper, A Study on Fourier Series and Fourier Laplace Transform are described, some results on differential operator are proved.

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