Extension Of Quasimetric On An Arbitrary Atomistic Lattice

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ABSTRACT

The present paper proposes to extend the notion of quasimetric on an arbitrary atomistic lattice L. Such a matric lattice on L will determine a neighbourhood (shortly nhd) structure, a generalisation of which leads to the concepts of uniformity as well as proximity on L.

Keywords : quasimetric lattice (non-symmetric metric), arbitrary atomistic lattice, uniformity, proximity

A set topology is a Kuratowskian Closure operator on the power set which is isomorphic to a complete Boolean lattice and which is also characterised in terms of quasimetric (non-symmetric metric). It is, therefore, possible to formulate a metric structure on such a lattice [1].

The present paper proposes to extend the notion of quasimetric on an arbitrary atomistic lattice L. Such a matric lattice on L will determine a neighbourhood (shortly nhd) structure, a generalisation of which leads to the concepts of uniformity as well as proximity on L. Throughout the section lattice L will be assumed to be atomistic unless and otherwise stated in which atoms will be denoted by $p, q, r, \ldots$

Definition 1. A quasimetric on an atomic lattice L is a positive real valued function defined on $L \times L$ such that

$M_1 : p = q \Rightarrow \partial(p, q) = 0$

$M_2 : \partial(p, r) \le \partial(p, q) + \partial(q, r)$

(L, ∂) is, then, called a quasimetric lattice.

Definition 2. If $x \ne 0$, then the distance function $\partial(x, p)$ on L is given by :

$$\partial(x, p) = \wedge_{q \le x} \{\partial(p, q)\}$$

The notion of closure operator is introduced as follows:

Definition 3. A closure operator (-) on L is a unary operator on L such that

$$c_1 : \overline{0} = 0$$

$$c_2 : \overline{x} = \vee_{\partial(x,p)=0} \rho, \qquad \text{if } x \ne 0$$

Theorem 1. (L, ∂) is a Kuratowskian topological space.

Proof. I. Let $p \le x$. Then by M, $\partial(x, p) = 0$ and consequently $p \le \overline{x}$.

Hence $x \le \bar{x}$.

II. $q \le \bar{\bar{x}} \Rightarrow \partial(\bar{x}, q) = 0$

$\Rightarrow$ for a given positive there exists

an atom p_1 such that $\partial(q, p_1) < \in$ for $p_1 \le \bar{x}$.

Similarly $p_1 \le \bar{x}$

$\Rightarrow$ for the same $\in$ there exists

an atom p_2 such that $\partial(p_1, p_2) < \in$ for $p_2 \le \bar{x}$.

Hence for a given $\in$ we find atoms p_1 and p_2 such that

$\partial(q, p_2) \le \partial(q, p_1) + (p_1, p_2) < 2\in$ for $p_2 \le x$. Hence $\partial(x, q) = 0$ and $q \le \bar{x}$. Hence $\bar{\bar{x}} \le \bar{x}$. But from the first part $\bar{x} \le \bar{\bar{x}}$. Hence $\bar{x} = \bar{\bar{x}}$.

III. $q \le \overline{x \vee y} \Rightarrow \partial(x \vee y, q) = 0$

$\Rightarrow \bigwedge_{p \le x \vee y} \partial(q, p) = 0$

$\Rightarrow$ there exists an atom $p_n \le x \vee y$ such that

$\partial(q, p_n) < \frac{1}{n}$ for all positive integers n

$\Rightarrow \wedge \partial(q, p_n) = 0 = \partial(x, q)$ or $\partial(y, q)$

$\Rightarrow$ either $q \le \bar{x}$ or $q \le \bar{y}$

$\Rightarrow q \le \bar{x} \vee \bar{y}$

$\Rightarrow \overline{x \vee y} \le \bar{x} \vee \bar{y}$

Next, $x < x \vee y \Rightarrow p \le x \Rightarrow p \le x \vee y$

$\partial(x, p) = 0 \Rightarrow \partial(x \vee y, p) = 0$

$\Rightarrow \bar{x} \le \overline{x \vee y}$

Similarly $\bar{y} \le \overline{x \vee y}$

Hence $\overline{x \vee y} \le \bar{x} \vee \bar{y}$. This prove the theorem.

Every quasimetric on L introduces the notion of a basic nhd structure on L.

Definition 4. Let $\in$ be any positive number. Then $\in$-nhd of p in L, i.e.,

$$n_p(\in) = \bigvee_{\partial(p,q)<\in} q$$

Theorem 2. The set $N = \{n_p(\in)\}$ is a basic nhd on L.

Proof. N_1 : By M_L we have $p \le n_p(\in)$

N_2 : $n_p(\in_1)$ and $n_p(\in_2)$ there exists a neighbourhood $n_p(\eta)$ such that $n_p(\eta) \le n_p(\in_1)$ and $n_p(\eta) \le n_p(\in_2)$, where $\eta = \min(\in_1, \in_2)$

N_3 : $q \le n_p(\in) \Rightarrow \partial(p, q) = \in - \eta$,

where $0 < \eta < \in$ and $q \ne p$.

$n_4(\eta) \le n_p(\in)$, for if $r \le n_q(\eta)$, then

$\partial(q, r) < \eta$ and by M_2 :

$\partial(p, r) \le \partial(p, q) + \partial(q, r) = \in - \eta + \eta = \in$

i.e., $r \le n_p(\in)$.

Theorem 3. If ∂ is symmetric on L, then (L, ∂) is a Hausdorff space.

Proof. Let $\partial(p, q) = 3\in > 0$. Then $p \ne q$ and $n_p(\in) \wedge n_p(\in) = 0$ for otherwise there would be an atom r such that $r \le n_p(\in)$ and $r \le n_q(\in)$ so that $\partial(r, p) < \in$ and $\partial(q, r) < \in$. Then by $M_2(p, q) < \partial(r, p) + \partial(r, q) < 2\in$ contracting the previous assumption.

In addition to a basic nhd structure there exist an open and a general a nhd structure [2]. Throughout the rest of the section we shall confine to a general nhd structure on an arbitrary atomistic lattice.

Definition 5. A general nhd $\overset{*}{N}$ on L is a set of all nhds of all atoms of L in which the following axioms, in addition to N_1 and N_3 of Theorem 2 hold :

N_2^* : If n_1 and n_2 are nhds of p, then $n_1 \wedge n_2$ is a nhd of p.

N_4: If n_p is a nhd of p and $m > n_p$, then m is a nhd of p.

$\overset{*}{N}$ is said to be symmetric iff $p < n_q \Rightarrow q < n_p$.

Definition 6. (L, N^*) is called a nhd lattice.

A closure of operator on (L, N^*) can be introduced as follows

Definition 7. The closure x of an element x in (L, N^*) is given by

$$\bar{x} = n_p \wedge x \overset{\vee}{>} o \ \ p \text{ for every nhd } n_p \in \overset{*}{N}.$$

Theorem 4. The closure operator (-) in (L, N^*) satisfies the following properties [3]

$$X < \bar{x}, \bar{o} = o, x < y \Rightarrow \bar{x} < \bar{y} \text{ and } \bar{\bar{x}} = \bar{x}.$$

Proof : I. Let $p < x$, $p < n_p \Rightarrow o < p < x \wedge n_p \Rightarrow p < \bar{x}$.

II. $\bar{o}$ is the sup of all atoms p such that $o \wedge n_p > o$, but no such atom p exists, whence $\bar{o} = o$.

III. $p < x < y \Rightarrow x \wedge n_p > o$

$\Rightarrow y \wedge n_p > o$

$\Rightarrow \bar{x} < \bar{y}$.

IV. $p \nless x \Rightarrow n_p \wedge x = o$ for some $n_p \in \overset{*}{N}$

$\Rightarrow n_p \wedge \bar{x} = o$ for some $n_p \in \overset{*}{N}$.

$\Rightarrow p \nleq \bar{\bar{x}}$.

Hence $\bar{x} \geq \bar{\bar{x}}$. But from the first part $\bar{x} \leq \bar{\bar{x}}$. Hence $\bar{x} \geq \bar{\bar{x}}$.

Theorem 5. The closure operator in a symmetric nhd lattice (L, N^*) is symmetric.

Proof. Let p and q be distinct atoms in L. Then $q < \bar{p} \Rightarrow p < n_q \Rightarrow q < n_p \Rightarrow p < \bar{q}$.

Theorem 6. A general nhd structure $\overset{*}{N}$ on L superimposes an open elemental structure by difining an element x to be open 1ff it includes a general nhd of every atom included in it.

Proof. I. o, $1 \in L \Rightarrow o$ and 1 are open.

II. x, y open and $p < x \wedge y \Rightarrow$ there exist n_p and m_p in $\overset{*}{N}$ such that $n_p < x$ and $m_p < y \Rightarrow n_p \wedge m_p < x \wedge y \Rightarrow x \wedge y$ is open.

III. If $\{x_\alpha : \alpha \in I\}$ is an open elemental set and $\underset{\alpha \in I}{\vee} X_\alpha$ exists,

Then $\underset{\alpha \in I}{\vee} X_\alpha$ is evidently a open element.

Theorem 7. The interior $\underline{x}$ of an element x in a nhd lattice is the sup of all atoms p such that $n_p < x$ for some $n_p \in \overset{*}{N}$.

Proof. Since every open element $z < x$ is contained in x, it suffices to prove that $\underline{x}$ is open. Now,

$p < x \Rightarrow n < x$ for some $n_p \in \overset{*}{N}$

$\Rightarrow L_q < m_q < x$ for every $q < m_q$, by Theorem 4 IV, for some L_q, $m_q < N^*$

$\Rightarrow m_q < \underline{x}$ for every $p < \underline{x}$

$\Rightarrow \underline{x}$ is open.

Remark 1. An interior operator in a general nhd lattice is distributive over $\wedge$.

For, $p < \underline{x} \wedge \underline{y} \Rightarrow p < \underline{x}, p < \underline{y}$

$\Rightarrow n_p < x, m_p < y$ for some $n_p, m_p \in \overset{*}{N}$

$\Rightarrow n_p \wedge m_p < x \wedge y$

$\Rightarrow p < \underline{x \wedge y}$

Uniformizations of and structure on L leads the following uniformity.

Definition 8. A uniform structure [4] on L is a family

$U = \{(u, L_o, L) : L_o$ is the set of all atoms of $L\}$ of mappings defined on L_o into L such that

$U_1 : p < u(p)$ for every $p \in L_o$, i.e., the identity function I defined by $I_{(p)} = p$ for all $p \in L_o$ is included in every function u.

$U_2 : v > u \in U \Rightarrow v \in U$, i.e., if $u \in U$ and $u(p) < v(p)$ for all $p \in L_o$, then $v \in U$ for all $p \in L_o$, then $v \in U$

$U_3 : u, v \in U \Rightarrow u \wedge v \in U$, where $u \wedge v$ denotes the function defined by $u \wedge v(p) = u(p) \wedge v(p)$ for every $p \in L_o$.

$U_4 : u \in U \Rightarrow$ there exists a $v \in U$ such that v o $v < u$ where v o v means $v(q) < u(p)$ for every $q < v(p)$.

(L, U) is called a uniform lattice which can be evident from the following theorem :

Definition 9. A function $u \in U$ defines a binary relation $\hat{u}$ on L_o by setting

$\hat{u} : \{(p, q) \quad p, q \in L_o$ and $q < u(p)\}$.

Theorem 8. $(L, \hat{u})$ is a uniform space consisting of a lattice and family.

$\hat{u} = \{\hat{u} : u \in u\}$ of relations on $L_o \subset L$

satisfying U_1 - U_4 and conversely, provided that L is closed with respect to arbitrary sup.

Proof. I. (i) $p < u(p) \Rightarrow I < u > \Rightarrow (p, p) \in \hat{I}$

$\Rightarrow (p, q) \in \hat{u} \Rightarrow \hat{I} \subset \hat{u}$

(ii) $\hat{v} \supset \hat{u} \in \hat{U} \Rightarrow (p,q) \in \hat{u}$

$\Rightarrow (p,q) \in \hat{u} \Rightarrow q < u(p)$

$\Rightarrow q < v(p) \Rightarrow v \in U \Rightarrow \hat{v} \subset \hat{U}$.

(iii) $\hat{u}, \hat{v} \in \hat{U} \Rightarrow u \wedge v \in U \Rightarrow (u \wedge v)^{\wedge} \in \hat{U}$

(iv) $v \,o\, v < u \Rightarrow \hat{v} o \hat{v} \subset \hat{u}$

II. Conversely, let $\hat{U}$ be a family of relations on L_o satisfying the usual conditions of uniformity. Then associating a relation *u* with a function *u* by defining

$u(p) = \bigvee_{(p,q) \in \hat{u}} q$, the set $\{\hat{u} : \hat{u} \in \hat{U}\}$ is easily verified to satisfy the conditions U_1 - U_4.

Separation axioms in a uniform lattice are also definiable [5].

Definition 10. A uniform lattice (L, U) is called :

T_o provide for distinct atoms *p* and *q* of *L* there exists a function $u \in U$ for which $p \nless u(q)$, or there exists a $v \in U$ such that $q \nless v(p)$;

T_1 provided for distinct atoms *p* and *q* there exist functions *u* and $v \in U$ for which $p \nless u(q)$ and $q \nless v(p)$;

T_2 provided for distinct atoms *p* and *q* there exist functions *u* and $v \in U$ for which $p \nless u(q)$, $q \nless v(p)$, $u(q)$ and $v(p)$ are disjoint elements.

Theorem 9. A uniform complete lattice is T_o iff $\bigwedge_{u \in U} u$ is an antisymmetric function,

T_1 iff $\bigwedge_{u \in U} u = I$ (identity function).

Proof. I. For $\wedge u(p)$ is antisymmetric iff $p \nless u(q)$ for distinct atom *p* and *q* and for some $u \in U$.

II. Since $I < u$, $I \wedge u$, it suffices to prove $\wedge u < I$. T_1-ness $\Rightarrow$ for distinct atom *p* and *q* i.e., $q \nless I(p)$, there exists a $u \in U$ such that $q \nless u(p)$, i.e., $u < I$, whence $\wedge u < I$.

Theorem 10. A symmetric function T_o-lattice is T_2-lattice.

Proof. I. T_0-ness $\Rightarrow$ for distinct atom *p* and *q*, $p \nless u(q)$, or $q \nless u(p)$ for some $u \in U$. Choose a symmetric function *v* such that $v \,o\, v < u$. Then $v(p)$ and $v(q)$ are distinct elements. If possible, let $r < v(p)$ and $r < v(q)$ whence $q < v(r)$ for symmetry, then $q < v(v(p)) < u(p)$ and by symmetry $p < v(v(p)) < u(q)$, which contradicts that $p \nless u(q)$ or $q \nless u(p)$.

A uniform structure on a sup complete atomistic lattice has its conjugate.

Definition 11. The function u^* on L_o into uniform sup complete lattice L defined by $u^*(p) = \vee_{q \in L_o} q$ for $p < u(q)$ is called the conjugate of u.

Theorem 11. The set $u^* : \{u^* : u \in U\}$ of conjugate functions of u in U is uniformity, called the conjugate uniformity.

Proof. I. $p < u(p) \Rightarrow p < u^*(p)$

II. $u^* \in U^*$ and $q < u^*(p) \Rightarrow p < u(q)$, $u \in U$. $q < u^*(p)$, $u^*(p) < v^*(p)$. $u^* \in U^* \Rightarrow p < u(q) < v(q) \Rightarrow v \in U \Rightarrow v^* \in U^*$

III. $u_1^*, u_2^* \in U^* \Rightarrow u_1, u_2 \in U \Rightarrow (u_1 \wedge u_2)^* \in U^*$

IV. $u^* \in U^* \Rightarrow u \in U \Rightarrow$ there exists a v such that v o $v < u \Rightarrow$ for $q < v(p)$, $r < v(q) \Rightarrow r < u(p)$. Hence v^* o $v^*(r) < u^*(r)$.

Remark 2. A uniform lattice is symmetric iff $U = U^*$.

Remark 3. T_1-ness property of a uniform lattice is conjugate invaritant, i.e., whenever a uniformity is T_1, then so in its conjugate.

Theorem 12. A T_1 uniform Boolean algebra has a T_1 open elemental structure, i.e. for which an antiatom h is an open elemental.

Proof. I. Let $q < h$. As the uniformity is T_1, for distinct atoms p and q, i.e. for $q \nless I(p) \wedge u < I$. Obviously, then there exists $u \in U$ such that $q \nless u(p)$ and $r < u(q) \Rightarrow r < h$, i.e. $u(q) < h$, whence is open.

As extension of uniformity U on an atomistic sup complete lattice L leads to a proximity structure on L.

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