REPRESENTATIONS BY THE FORM WITH ODD PRIME INVARIANTS

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Abstract : This paper deals with the representation by the quadratic form in three variables with odd prime invariants. In this paper a primitive quadratic form over the field of integers with odd invariants is considered and another form mutually primitive to it. Then it is proved that the number of representations by form is greater than the number of classes of integral primitive binary quadratic forms.

***Index Terms** - Quadratic Forms, Primitive, Representations, Primes and Odd Invariants.*

I. INTRODUCTION

Representation theory plays very important role in the field of mathematics. In the paper (Chetna and Singh [3]), the general hypothesis given by Riemann is considered and found the number of integers represented by forms in three variables with small determinants. This paper deals with the representation by the quadratic form in three variables with odd prime invariants. In this paper following lemmas are used to obtain the desired result.

Lemma 1[9]: Let f be a positive quadratic form in three variables over the field of integers with the determinant δ and let equations are

$l + L_i = V_i K_i \quad i = 1, \dots, n \qquad (1)$

where l is an integer, $L_1, \dots\dots, L_n$ are the integral forms with norms m, $K_1, \dots\dots, K_n$ are the proper integral of norm k prime to 2δ, $V_1, \dots\dots, V_n$ are the integral of norm v prime to k. Let the inequalities be

$n > x_1 m^{\frac{1}{2}-\epsilon} \qquad (2)$

$x_2 m^{\sigma-\epsilon} \le k \le x_3 m^{\sigma+\epsilon} \qquad (3)$

$\gcd(m,k) < x_4 m^{\epsilon} \qquad (4)$

where $0 < \sigma \le \frac{1}{2}$ are the real numbers and for $x_i > 0, i = 1,2,3,4$ there exist constant $\in > 0$, where $\in$ is an arbitrary real number. Then, the number w among distinct integrals $K_1, \dots\dots, K_n$ is given by

$w > x_\epsilon m^{\sigma-3\epsilon} \qquad (5)$

where $x_\epsilon > 0$ constant depending only on $\epsilon, u, x_1, x_2, x_3, x_4$.

Lemma 2[9]: Let f is the positive integral quadratic form in three variables of determinant δ and let the equations be

$l + L_i = V_i K_i, \quad (i = 1, \dots\dots, n) \quad (6)$

where l is an integer, $L_1, \dots\dots, L_n$ are the different primitive integrals forms with norms m, $K_1, \dots\dots, K_n$ are the integral forms with norms k prime to 2δ, $V_1, \dots\dots, V_n$ are the integral forms with norm v prime to k. Let $m = m_1 m_2$, where $m_1$be square of an integer and m_2 be square-free and

$m_1 < x_{17} e^{\sigma_{17}\frac{\sqrt{\log m}}{\log(\log m)}} \quad (7)$

Let the inequalities

$n > x_{18} m^{\frac{1}{2}} e^{-\sigma_{18}\frac{\sqrt{\log m}}{\log(\log m)}} \quad (8)$

$x_{19} m^{\mu} e^{-\sigma_{19}\frac{\sqrt{\log m}}{\log(\log m)}} \le k \le x_{20} m^{\mu} e^{\sigma_{20}\frac{\sqrt{\log m}}{\log(\log m)}} \quad (9)$

$\gcd(m,k) < x_{21} e^{\sigma_{21}\frac{\sqrt{\log m}}{\log(\log m)}} \quad (10)$

$\tau(k) < x_{22} e^{\sigma_{22}\frac{\sqrt{\log m}}{\log(\log m)}} \quad (11)$

where μ be a real number, $0 < \mu \le \frac{3}{8}, x_{17}, x_{18}, x_{19} > 0, x_{20} x_{21} x_{22} \sigma_{17} \sigma_{18} \sigma_{19} \sigma_{20} \sigma_{21} \sigma_{22}$ are constants that depend only on δ. Suppose that for different $K_1, \dots\dots, K_n$ there exist distinct$w$, then

$w > x m^{\mu} e^{-\sigma\frac{\sqrt{\log m}}{\log(\log m)}} \quad (12)$

where $\sigma, x > 0$ are the constants that depend only on δ.

II. REPRESENTATION OF INTEGERS BY INTEGRAL QUADRATIC FORM IN THREE VARIABLES WITH ODD PRIME INVARIANTS

Now by using the above lemmas the following theorem gives on the representation by positive quadratic form in three variables with odd prime invariants.

Theorem: Let $f = f(x_1, x_2, x_3)$ be an integral primitive quadratic form in three variables with odd invariant $[d,k]$, $F = F(x_1, x_2, x_3)$ be a mutual primitive form of it with invariant $[d,k]$. Let R is a primitive form with norms $r > 1$ relatively prime to $2dk$, m is a positive integer relatively prime to r such that $f(x_1,x_2,x_3) \equiv \mathrm{m}(\mathrm{mod}\ 2^3\mathrm{d}^2\mathrm{km})$. Let l is an integer satisfying the congruence

$l^2 + km \equiv 0m(mod\ r)$ (13)

Let h is an integer relatively prime to $2dkr$ and LQ is a primitive form on (mod h) with the condition $N(L_0) \equiv km(mod\ h)$. Finally, suppose that $\Delta_{f,m}$ be the field for the form $f(x_1,x_2,x_3) = m$ in three variables with the condition $\gamma > 0$. We denote $r_{h,L_0}(\Delta_{f,m}, R, l)$ as the number of primitive integral form L with norms km such that $L \equiv L_0(mod\ h), \frac{l+L}{R}$, where $L \in \Delta_{f,m}$. Then for $m \geq m'$, we have

$r_{h,L_0}(\Delta_{f,m}, R, l) > xg(-km)$ (14)

where $g(-km)$ are the actual number of classes of integral primitive binary quadratic forms with positive determinant km where $m', x > 0$ are the constants that depend only on d, k, r, h, γ, u in the field $\Delta_{f,m}$.

Proof: Let us suppose that $s = r^t$. By (Chetna and Singh [3]), we can say that $t = t(d,k)$ is the number of primitive form L with norms kms^2 is less than $x_1 g(-kms^2)$, where the constant $x_1 > 0$ depends only on d and k. Among these forms there are

$g' > x_2 g(-kms^2)$ (15)

equivalent to each other, where the constant $x_2 > 0$ depends only on d and k. Let us consider

$L_1, \ldots\ldots\ldots, L_{g'}, g' > x_2 g(-kms^2)$ (16)

We show that for sufficiently large m, where $m > m'$ a set of (d,k) in (16) can be chosen such that $g > x_3 g(-kms^2)$ for primitive forms $L_1, \ldots\ldots\ldots, L_n$ with norms kms^2 and have the equations $sl' + L_i = V_i M_i$, $(i = 1, \ldots\ldots\ldots, n, g > x_3 g(-kms^2))$,where $M_1, \ldots\ldots\ldots, M_g$ are the integral forms with norms r^w, $V_1, \ldots\ldots\ldots, V_g$ are the integral forms with norms v prime to r and for integer w following inequality occurs $x_4 m^\mu \leq r^w < m^\mu$, where μ is the real number such that $0 < x_5 \leq \mu \leq \frac{2}{2}$ and the constants $x_3 > 0, x_4 > 0$, and $x_5 > 0$ depend only on d, k . Here, we assume that the number m is so large such that $m \geq m'^{(1)}(d,k)$ and for $w > 0$ the integer l' satisfies the congruence $l' \equiv l(mod\ r)$. Let $a = \left[\frac{1}{x_2}\right] + 1$ is a positive integer depending only on d, k. Then, by (15) we have $g' > \frac{g(-kms^2)}{a}$. We define an integer e for the inequalities

$\frac{1}{r} m^{\frac{1}{8a}} \leq r^e < m^{\frac{1}{8a}}$ (17)

and consider integers, $z_0 = r^{2ae}, z_1 = r^{(2a+1)e}, \ldots\ldots\ldots, z_y = r^{(2a+a)e} = r^{3ae}$. Since the number m is so large such that $m \geq m'^{(2)}(d,k,r)$, therefore by (Kane [4]),we have $ea > t$. For each z_i we consider an integer l_i satisfying the following conditions:

$$\gcd\left(\frac{(sl_i)^2 + kms^2}{z_i}, r\right) = 1, \quad (i = 1, \ldots\ldots\ldots, y) \quad (18)$$

and by condition (13) we can find $z_i \leq z_a = r^{3ae} < m^{\frac{3}{8}}, u_i > \frac{m}{z_a} > m^{\frac{5}{8}}, z_i \leq u_i$ $(i = 0,1, \ldots\ldots, a)$. Now, consider the primitive classes of positive binary quadratic forms with the determinant kms^2, where $\Phi_j \Theta_\lambda (j = 1, \ldots\ldots\ldots, g'; \lambda = 0,1, \ldots\ldots\ldots, a)$. Therefore, by (Niven et all[5]) there exist fixed pair (λ_0, δ_0) for which we have

$> \frac{x_2}{(a+1)^2} g(-kms^2)$ (19)

with the condition $\Phi_i^{-1}\Phi_j = \Theta_{\lambda_0}\Theta_{\delta_0}^{-1}$ (20)

where $\Phi_i^{-1}\Phi_j$ is the class by the pair (L_i, L_j). By (20) in this class, there is a binary quadratic form (v, sl', r^2) where v relatively prime to $r$and l' satisfies the equation (7) and $w = e(\lambda_0 - \delta_0)$. Let c be a four-dimensional $(x_0^2 + kf)$ corner in the field. Then by (Chetna and Singh [2]), we have $\frac{c}{2\pi^2} = \frac{\gamma''}{4\pi}$ (21)

is the form in the field Ω_W depending only on the form in the region $\Delta''_{f,m}$ and it depends on W and finally on the form in the field $\Delta_{f,m}$. Consider A with condition that $AL \equiv L_0 A$, $N(A) \equiv r^a(mod\ h)$. We choose a primitive form S with norms r^z where z is a constant depending only on d, k, h, γ in the field $\Delta_{f,m}$ such that $S = RS_{xu}R \ldots\ldots\ldots RS_{1u}R \ldots\ldots\ldots RS_{11}$

for any i and j $(1 \leq i \leq n, 1 \leq j \leq u)$. If L is a primitive form with norm km in the field Ψ_i with $L^{(j)}(mod\ h)$, then

$$\begin{cases} (S_{ij}R \ldots\ldots\ldots RS_{11})L(S_{ij}R \ldots\ldots\ldots RS_{11})^{-1} \in \Delta_{f,m} \\ (S_{ij}R \ldots\ldots\ldots RS_{11})L \equiv L_0(S_{ij}R \ldots\ldots\ldots RS_{11})(mod\ h) \end{cases} \quad (22)$$

Firstly, consider the form S_{11} with norm $r^{a_1 - 1}$ where a_1 is bounded above by a constant depending only d, k, h, γ in the field $\Delta_{f,m}$ for which the product RS_{11} is primitive and if a primitive form L with norm km in Ψ_1 with $L^{(1)}(mod\ h)$, then $\begin{cases} S_{11}LS_{11}^{-1} \in \Delta_{f,m} \\ S_{11}L \equiv L_0 S_{11}(mod\ h) \end{cases}$. Now, we choose the number $a_1 = a_1(d,k,h,\Delta_{f,m})$ so large such that there exist a primitive form T_1 with norms r^{a_1} with the following properties: (a) R divides T_1, $(b)\ T_1$ belongs to $\Lambda_{L^{(1)}}^{(a_1-1)}$, $(c)\ T$ belongs to Λ_{W_1}. Let us suppose that $T_1 = RS_{11}$, where $S_{11}$is a primitive form, then

$$\begin{cases} (S_{12}RS_{11})L(S_{12}RS_{11})^{-1} \in \Delta_{f,m} \\ (S_{12}RS_{11})L \equiv L_0(S_{12}RS_{11})(mod\ h) \end{cases}$$

By (Chetna and Singh [3]), we consider the number $a_1 = a_1(d,k,h,\Delta_{f,m})$ so large such that there is a primitive form T_2 with norms r^{a_2+1} with properties:

a) R divides T_1, $(b)\ T_1$ belongs to $\Lambda^{(a_2)}_{S_{11}L^{(1)}(S_{11})^{-(mod\ h)}}$, (c) T belongs to $\Lambda_{S_{11}W_1 S_{11}^{-1}}$

Thus we can deduce the following proposal Γ at $m \geq m'$where m' is a constant depending on d, k, h, γ in the field $\Delta_{f,m}$ for some integer $\tau, c - z - l \geq \tau \geq l$, by (Shimura [7]), there is number greater than $x_6 g(-kmc^2)$ and equation

$sl' + L = VM, M = A^{(\tau)}SC^{(\tau)}, N(C^{(\tau)}) = r^{\tau}$ (23)

where $A^{(\tau)}$ and $C^{(\tau)}$ are primitive forms and $x_6 > 0$ is a constant depending on d, k, h, γ in the field $\Delta_{f,m}$.This is proved by contradiction. Let Γ does not exist and we consider an integer $\tau_0 \geq z$ and consider the set of indices $\{\tau_1, \tau_2, \dots \dots \dots, \tau_{s_1}\}$ where

$\tau_{s_1} = \tau_{s_{1-1}} + \tau_0 \leq c - z - l, \tau_{s_1} + \tau_0 > c - z - l$ (24)

We consider the number m so large such that $s_1 \geq 1$. By (24) and the inequality (6), we have

$$x_7 \log m \geq x_8 \log m \qquad (25)$$

where the constants $x_7 > 0, x_8 > 0$ and $x_9 > 0$ depend on d, k, h, γ in the field $\Delta_{f,m}$. By our assumption for any real number $\lambda > 0$ depending on d, k, h, γ over the field $\Delta_{f,m}$, there exist increasing sequence of numbers m upto infinity such that for every $\tau \geq l$ there exist a number of indices $i, (1 \leq i \leq g)$ by (Burton [1]) with the condition

$M_i = A_i^{(\tau)}SC_i^{(\tau)}, N(C_i^{(\tau)}) = r^{\tau}$ (26)

where $A_i^{(\tau)}$ and $C_i^{(\tau)}$ are the primitive form which is $< \lambda g$. From (5) we take equation

$sl' + L_i = V_i M_i, i = 1, \dots \dots \dots, g_1$ (27)

Without any loss of generality, we can change the indexing, for which we have the following property that for each $i, (i = 1, \dots \dots \dots, g_1)$ the number of indexes $\tau \in [\tau_1, \tau_2, \dots \dots \dots, \tau_{s_1}]$ with the condition (26) is less than $2\lambda s_1$. Now we show that

$g_1 > \frac{x_3}{2} g(-kms^2)$ (28)

Let v_1 be the total number of primitive form with norms r^s with the following property that the number of indexes $\tau \in [\tau_1, \tau_2, \dots \dots \dots, \tau_{s_1}]$ for which $M = A^{(\tau)}SC^{(\tau)}, N(C^{(\tau)}) = r^{\tau}$, where $A^{(\tau)}$ and $C^{(\tau)}$ are the primitive forms. Now, we show that we can choose $\tau_0 > 0$ and $\lambda > 0$ depending on d, k, h, γ in the field $\Delta_{f,m}$, then $v_1 < x_\sigma m^{\mu - \sigma}$, where $\sigma > 0$ and $x_\sigma > 0$ are the constants depending on d, k, h, γ over the field $\Delta_{f,m}$. Let τ and τ' positive integers such that$\tau - \tau' \geq \tau_0 \geq z$. Consider a fixed class O equivalent to the primitive form over the$(mod\ r^{\tau'})$. Then:

1) the number of classes similar to primitive form over the$(mod\ r^{\tau})$ with the provision that each class contains a primitive form with norms r^{τ} is divisible by S and is equal to

$\rho r^{\tau} \prod \left(1 + \frac{1}{p}\right) \times (1 + \theta)$ (29)

2) the number of other classes similar to primitive form over the $(mod\ r^{\tau})$ with the condition

$(1 - \rho) r^{\tau} \prod \left(1 + \frac{1}{p}\right) \times (1 + \theta)$ (30)

Here $\theta \to 0$ if $r^{\tau} \to \infty$ and d and k are fixed.

3) the number of classes similar to primitive form over the $(mod\ r^{\tau})$ with the condition that each class contains a primitive form with norms r^{τ} is divisible by S and for fixed class O is equal to

$\rho r^{\tau - \tau'} \times (1 + \theta)$ (31)

4) the number of other classes similar to primitive form over the $(mod\ r^{\tau})$ in the fixed class O is equal to

$(1 - \rho) r^{\tau - \tau'} \times (1 + \theta)$ (32)

Here $\theta \to 0$ if $r^{\tau - \tau'} \to \infty$, and d and k are fixed.

Further, by (Burton[1]) (29) is a direct significance of the observations when one considers that two primitive form $A_1$and A_2 with norms r^{τ} similar to $(mod\ r^{\tau})$ if and only if there exist form unit E with the condition $A_1 = EA_2$. The condition (30) follows from the form (29). The form (31) follows by (Timothy [8]) and (32) follows from (31). Then, by using (32) and (30), we obtain

$$v(r^s, \tau'_{s'_1}, \dots \dots \dots, \tau'_1) = r^s \prod \left(1 + \frac{1}{p}\right) \times (1 - \rho)^{s'_1} \times \prod_{i=1}^{s'_1} (1 + \theta_i) \qquad (33)$$

Further, we have

$$\left(\log \frac{3}{2} m\right) \left(m^{\beta_1 \log \beta_1 - \beta_2 \log \beta_2 - (\beta_1 - \beta_2) \log(\beta_1 - \beta_2) + (\beta_1 - \beta_2) \log\left(1 - \frac{\rho}{2}\right)}\right) \leq (x_{12} m^{-\sigma}) \quad (34)$$

Using the inequality (Oh [6]) $g(-m) > x_\varepsilon m^{\frac{1}{2} - \varepsilon}$, where $\varepsilon > 0$ is an arbitrary real number, $x_\varepsilon > 0$ is the constant depending only on ε. From this and from (28) we deduce that $g_1 > x'_\varepsilon m^{\frac{1}{2} - \varepsilon}$. Therefore, if v is the number of different form M_i in (27) then by Lemma 1, we have $v > x''_{\varepsilon'} m^{\mu - \varepsilon'}$, where $\varepsilon' > 0$ is a positive real number and $x''_{\varepsilon'} > 0$ is the constant depending only on $\varepsilon', d, k, h, \gamma$ in the field $\Delta_{f,m}$. Each of these equations corresponds to equation

$sl' + L' = C^{(\tau)}(VA^{(\tau)})S = C^{(\tau)} L (C^{(\tau)})^{-1}$ (35)

where L' is the form with norm kms^2. Since$\overline{C^{(\tau)}} L' \equiv 0 (mod\ s)$, then (Shimura [7])

$L' \equiv 0 (mod\ s), L' = sL''$ (36)

where L'' is the form with norm km. Thus, the set of primitive form L with norm kms^2 through (35) and (36) mapped into a set of primitive formL'' with norm km. Each form corresponds to L'' be less than equal to x_{13}. So we get value greater than $x_{15} g(-km)$ and the equation

$L''_i \equiv L^{(\xi_0)} (mod\ h)$ (37)

is equal to $g_3 > x_{16} g(-km)$, where the constant $x_{16} > 0$ depending only on k, d, r, h, γ over the field $\Delta_{f,m}$. Now, we consider $S = S_2 R S_1$ where $S_1 = S_{\xi_0 \xi_0}{}' R \dots \dots \dots R S_{11}$ from (37), we deduce that

$$r_{h, L_0}(\Delta_{f,m}, R, l) > x_{16} g(-km)$$

for primitive integral form L'''_i, which follows the proof.

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