New Subclass of Bi-univalent Functions Subordinate To Chebyshev Polynomials

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Abstract: In the present investigation, we introduce a new subclass of bi-univalent functions by subordinating to Chebyshev polynomials and find the upper bounds for first two coefficients and Fekete-Szegö functional for the functions in this class.

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1. Introduction and definitions

Let U be the class of bounded functions

$$u(z)=\sum_{k=1}^{\infty}b_k z^k$$

which are analytic in the unit disc $E=\{z:|z|<1\}$ and satisfying the conditions $u(0)=0$ and $|u(z)|<1$.

Let A be the class of analytic functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k. \tag{1.1}$$

Further, let S denote the class of functions in A which are univalent in E.

It is well known that every function $f\in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z \qquad (z\in E)$$

and

$$f\left(f^{-1}(w)\right)=w \quad \left(|w|<r_0(f):r_0(f)\geq \frac{1}{4}\right)$$

where

$$f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\dots \tag{1.2}$$

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z)\in U$, such that $f(z)=g(u(z))$. This result is known as principle of subordination.

A function $f\in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E.

Let Σ denote the class of bi-univalent functions in E given by (1.1).

Lewin [7] investigated this class Σ and obtained the bound for the second coefficient of the bi-univalent functions. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example [1,4,6-11]).

Chebyshev polynomials which we are going to use in this work played an important role in applied mathematics, numerical analysis and approximation theory. There are four types of Chebyshev polynomials but the majority of research work dealing with orthogonal polynomials of Chebyshev family, contain mostly results of Chebyshev polynomials of first and second kind $T_n(x)$ and $U_n(x)$. The Chebyshev polynomials of first and second kinds are orthogonal for $t\in[-1,1]$ and are defined as below:

Definition 1.1 The Chebyshev polynomials of the first kind are defined by the following recurrence relation:

$$T_0(t)=1,\quad T_1(t)=t,\quad T_{n+1}(t)=2tT_n(t)-T_{n-1}(t).$$

The generating function for the Chebyshev polynomials of first kind is given by:

$$F(z,t)=\frac{1-tz}{1-2tz+z^2}=\sum_{n=0}^{\infty}T_n(t)z^n.$$

Definition 1.2 The Chebyshev polynomials of the second kind are defined by the following recurrence relation:

$$U_0(t)=1,\quad U_1(t)=2t,\quad U_{n+1}(t)=2tU_n(t)-U_{n-1}(t).$$

The generating function for the Chebyshev polynomials of second kind is given by:

$$H(z,t)=\frac{1}{1-2tz+z^2}=\sum_{n=0}^{\infty}U_n(t)z^n.$$

The Chebyshev polynomials of the first and second kind are connected by the following relations:

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t); \qquad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

Definition 1.3 For $0 \le \lambda \le 0$ and $t \in (-1,1)$, a function $f(z) \in A$ is said to be in the class $M'_{\Sigma}(\lambda, t)$ if the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} \prec H(z,t) = \frac{1}{1 - 2tz + z^2} \tag{1.3}$$

and

$$\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda) g(w) + \lambda w g'(w)} \prec H(w,t) = \frac{1}{1 - 2tw + w^2}, \tag{1.4}$$

where the function $g(w) = f^{-1}(w)$ is defined by (2.2).

Also several subclasses of bi-univalent functions subordinated to Chebyshev polynomials were studied by various authors (see [2], [4], [11])

In this paper, we use the Chebyshev polynomial expansions to obtain estimates for the initial coefficients $|a_2|$ and $|a_3|$ for the functions in the class $M'_{\Sigma}(\lambda, t)$. We also solve Fekete-Szegö problem for functions in this class.

2. Coefficient bounds for the function class $M'_{\Sigma}(\lambda, t)$

Theorem 2.1 If $f(z) \in M'_{\Sigma}(\lambda, B)$, then

$$|a_2| \le \frac{2t\sqrt{2t}}{\sqrt{\left|(1+\lambda)^2 - 4t^2\lambda^2\right|}}$$

(2.1)

and

$$|a_3| \le \frac{4t^2}{(1+\lambda)^2} + \frac{t}{(1+2\lambda)}. \tag{2.2}$$

Proof. As $f(z) \in M'_{\Sigma}(\lambda, B)$, from (1.3) and (1.4), we obtain

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} = 1 + U_1(t) w(z) + U_2(t) w^2(z) + \ldots$$

(2.3)

and

$$\frac{wg'(w)+\lambda w^2 g''(w)}{(1-\lambda)g(w)+\lambda wg'(w)} = 1+U_1(t)v(w)+U_2(t)v^2(w)+...$$

(2.4)

for some analytic functions where

$$w(z)=c_1 z+c_2 z^2+c_3 z^3+...$$

and

$$v(w)=d_1 w+d_2 w^2+d_3 w^3+...$$

such that $w(0)=v(0)=0, |w(z)|<1$ and $|v(w)|<1.$

It follows from (2.3) and (2.4) that

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)} = 1+U_1(t)c_1(z)+\left[U_1(t)c_2+U_2(t)c_1^2\right]z^2+...$$

(2.5)

and

$$\frac{wg'(w)+\lambda w^2 g''(w)}{(1-\lambda)g(w)+\lambda wg'(w)} = 1+U_1(t)d_1(w)+\left[U_1(t)d_2+U_2(t)d_1^2\right]w^2+...$$

(2.6)

On equating the coefficients of z and z^2 in (2.5) and of w and w^2 in (2.6), we get

$$(1+\lambda)a_2 = U_1(t)c_1,$$

(2.7)

$$2(1+2\lambda)a_3 - a_2^2(1+\lambda)^2 = U_1(t)c_2+U_2(t)c_1^2,$$

(2.8)

and

$$-(1+\lambda)a_2 = U_1(t)d_1,$$

(2.9)

$$-2(1+2\lambda)a_3 + a_2^2\left[3(1+2\lambda)-\lambda^2\right] = U_1(t)d_2+U_2(t)d_1^2,$$

(2.10)

(2.7) and (2.9) together gives

$$c_1 = -d_1$$

(2.11)

and

$$2(1+\lambda)^2 a_2^2 = U_1^2(t)(c_1^2 + d_1^2).$$

(2.12)

Adding (2.8) and (2.10), we get

$$2(1+2\lambda)a_2^2 = U_1(t)(c_2 + d_2) + U_2(t)(c_1^2 + d_1^2).$$

(2.13)

Using (2.12) in (2.13), we obtain

$$\left[2(1+2\lambda) - \frac{2U_2(t)}{U_1^2(t)}(1+\lambda)^2\right]a_2^2 = U_1(t)(c_2 + d_2).$$

(2.14)

It is well known [ ] that if $|w(z)|<1$ and $|v(w)|<1,$ then $|c_j|\le 1$ and $|d_j|\le 1$ for all $j \in N.$ (2.15)

Also it is obvious from definition (1.2) that

$$U_1(t) = 2t \text{ and } U_2(t) = 4t^2 - 1. \tag{2.16}$$

So (2.14) gives (2.1).

Now subtracting (2.10) from (2.8), we get

$$4(1+2\lambda)a_3 - 4(1+2\lambda)a_2^2 = U_1(t)(c_2 - d_2) + U_2(t)(c_1^2 - d_1^2).$$

(2.17)

Using (2.11) in (2.17), it yields

$$a_3 = a_2^2 + \frac{U_1(t)(c_2 - d_2)}{4(1+2\lambda)}.$$

(2.18)

Using (2.12), (2.15) and (2.16) in (2.18), we obtain (2.2).

3. Fekete-Szegö Problem for the function class $M'_\Sigma(\lambda,t)$

Theorem3.1 If $f(z) \in M'_\Sigma(\lambda, B),$ then for some $\mu \in R,$

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2t}{1+2\lambda}; |h(\mu)| \le \dfrac{1}{4(1+2\lambda)} \\ 8|h(\mu)|t; |h(\mu)| \ge \dfrac{1}{4(1+2\lambda)} \end{cases} \tag{3.1}$$

where

$$h(\mu)=\frac{U_1^2(t)(1-\mu)}{2\left[(1+2\lambda)U_1^2(t)-(1+\lambda)^2U_2(t)\right]}.$$

Proof. Using (2.14) and (2.18), we get

$$a_3-\mu a_2^2=a_2^2+\frac{U_1(t)}{4(1+2\lambda)}(c_2-d_2)-\mu\frac{U_1^3(t)(c_2+d_2)}{2(1+2\lambda)U_1^2(t)-2(1+\lambda)^2U_2(t)}$$

$$=\frac{U_1(t)}{4(1+2\lambda)}(c_2-d_2)+(1-\mu)\left[\frac{U_1^3(t)(c_2+d_2)}{2(1+2\lambda)U_1^2(t)-2(1+\lambda)^2U_2(t)}\right]$$

$$=U_1(t)\left[\left(h(\mu)+\frac{1}{4(1+2\lambda)}\right)c_2+\left(h(\mu)-\frac{1}{4(1+2\lambda)}\right)d_2\right] \tag{3.2}$$

where $h(\mu)=\frac{U_1^2(t)(1-\mu)}{2\left[(1+2\lambda)U_1^2(t)-(1+\lambda)^2U_2(t)\right]}.$

Hence (3.1) can be easily obtained from (3.2).

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