SOLUTION OF DIFFUSION EQUATION WITH CONSTANT CO-EFFICIENT IN CYLINDRICAL AND SPHERICAL COORDINATES

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Abstract: This paper aims to apply the variables separation Method to solve the three-dimensional Diffusion equation with constant coefficient in cylindrical and spherical coordinates. Illustrative some examples are related to known results.

Keywords: cylindrical coordinates, spherical coordinates

Basic definitions:

The diffusion equation in one dimensional:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2},$$

On the interval $x \in [0, L]$ *with initial condition*

$$u(x,0) = f(x), \qquad \forall x \in [0, L]$$

And dirichlet boundary condition

$$u(0,t) = u(L,t) = 0 \quad \forall t > 0$$

The diffusion equation in two dimensional:

$$\frac{\partial u}{\partial t} = D(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})$$

Where $u = u(x,y,t), x \in [a_x, b_x], y \in [a_y, b_y]$.the second-order derivative in space leads to a demand for two boundary conditions.

The diffusion equation in three dimensional:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

Where T is a temperature and α is a diffusion coefficient.

Bessel differential equation:

The Bessel differential equation is the linear second-order ordinary differential equation given by

$$x^2 \frac{d^2 y}{dx^2}+x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

SOLUTION OF DIFFUSION EQUATION IN CYLINDRICAL CO-ORDINATES

Consider a three-dimensional diffusion equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

Where T is a temperature and α is a diffusion coefficient

In cylindrical co-ordinates (r,θ,z) the above equation becomes

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \quad (1.1)$$

Where T=T(r,θ,z,t)

Let us assume separation of variables in the form T(r,θ,z,t) =R(r) H(θ)Z(z) β(t)

Substitute in equation (1.1), it becomes

$$R''HZ\beta + \frac{1}{r}R'HZ\beta + \frac{1}{r^2}H''RZ\beta + Z''RH\beta = \frac{\beta'}{\alpha}RHZ$$

Dividing by RHZ β

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \frac{Z''}{Z} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2 \text{ (say)}$$

Where $-\lambda^2$ is a separation constant. Then

$$\beta' + \alpha\lambda^2\beta = 0 \quad (1.2)$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 = -\frac{Z''}{Z} = -\mu^2 \text{(say)}$$

where μ^2 is a separation constant. Then

$$Z'' - \mu^2 Z = 0 \quad (1.3)$$

Thus the equation determining Z,R and H becomes

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 + \mu^2 = 0$$

(or)

$$r^2\frac{R''}{R} + r\frac{R'}{R} + +(\lambda^2 + \mu^2)r^2 = -\frac{H''}{H} = v^2 \text{ (say)}$$

Therefore,

$$H'' + v^2 H = 0 \quad (1.4)$$

$$R'' + \frac{1}{r}R' + [(\lambda^2 + \mu^2) - \frac{v^2}{r^2}]R = 0 \quad (1.5)$$

Equation (1.2)-(1.4) have particular solutions of the form

$\beta = e^{-\alpha\lambda^2 t}$

H=Ccos $v\theta + \sin v\theta$

Z=A$e^{\mu z}$+$Be^{-\mu z}$

The diffusion equation (1.5) $R'' + \frac{1}{r}R' + [(\lambda^2 +)\mu^2 - \frac{v^2}{r^2}]$R=0 is called bessel's equation of order v and its general solution is

$$R(r)=C_1 J_V(\sqrt{(\lambda^2 + \mu^2 r)}) + C_2\, y_V(\sqrt{(\lambda^2 + \mu^2 r)})$$

Where J_V (r) and y_V (r) are Bessel functions of order v of the first and second kind, respectively.

Equation (1.5) is singular when r=0.The physically meaningful solutions must be twice continuously differentiable in $0 \le r \le a$

Hence, Equation (1.5) has only one bounded solution,

$$R(r)= J_V(\sqrt{(\lambda^2 + \mu^2 r)})$$

Finally, the general solution of the equation (1.1) is given by

$$T(r,\theta,z,t) = e^{-\alpha\lambda^2 t}[Ae^{\mu z}+Be^{-\mu z}]\,[\,C\cos v\theta + D\sin v\theta\,]\; J_V(\sqrt{(\lambda^2 + \mu^2 r)}).$$

EXAMPLE:1

Determine the temperature T(r,t) in the infinite cylinder $0 \le r \le a$ when the initial temperature is T(r,0)= f(r) and the surface r=a is maintained at 0° temperature.

Solution:

The governing PDE from the data of the problem is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

Where T is a function of r and t only. Therefore

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (1.6)$$

The corresponding boundary and initial conditions are given by

Boundary condition: T(a,t)=0

Initial condition : T(r,0)= f(r)

The general solution of equation (1.6) is

$$T(r,t)=A\, exp[-\alpha\lambda^2 t]\, J_0(\lambda r)$$

Using the boundary condition, we obtain

$$J_0(\lambda a)=0$$

Which has an infinite no. of roots, ε_n (n=1,2,3…∞).Thus,we get from the superposition principle The equation is

$$T(r,t)=\sum_{n=1}^{\infty} A_n \exp(-a\varepsilon_n^2 t)\, J_0(\boldsymbol{\varepsilon_n} r)$$

Using initial condition T(r,0)= f(r) we get,

$$f(r)= \sum_{n=1}^{\infty} An\, J_0(\boldsymbol{\varepsilon_n} r)$$

to compute A_n multiply both sides by r $J_0(\varepsilon_m r)$ and integrate with respect to r

$$\int_0^a r f(r)\, J_0(\varepsilon_m r) dr = \sum_{n=1}^{\infty} A_n \int_0^a r\, J_0(\varepsilon_m r)\, J_0(\varepsilon_n r)\, dr$$

$$= \begin{cases} 0 & \text{for } n \neq m \\ A_m(\frac{a^2}{2})a^2\, J_1^2(\varepsilon_m a) & \text{for } n=m \end{cases}$$

Which gives

$$A_m = \frac{2}{a^2\, J_1^2(\varepsilon_m a)} \int_0^a u\, f(u)\, J_0(\varepsilon_m u) du$$

Hence the final solution of the problems is

$$T(r,t) = \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_0(\varepsilon_m r)}{J_1^2(\varepsilon_m a)} \exp(-\alpha \varepsilon_m^2 t)[\int_0^a u\, f(u) J_0(\varepsilon_m u)\, du]$$

SOLUTION OF DIFFUSION EQUATION IN SPHERICAL COORDINATES

Consider a 3-D diffusion equation,

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

Let T=T(r,θ,ϕ,t)

This equation can be written in spherical coordinates,

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial T}{\partial \theta}) + \frac{1}{r^2 sin^2(\theta)}\frac{\partial^2 T}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (2.1)$$

This equation is separated by assuming the temperature function of the form

$$T = R(r)H(\theta)\, \Phi(\phi)\beta(t) \quad (2.2)$$

Substituting (2.2) in (2.1),we get

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2 \sin\theta}\frac{1}{H}\frac{d}{d\theta}(\sin\theta \frac{dH}{d\theta}) + \frac{1}{\Phi r^2 sin^2(\theta)}\frac{d^2\Phi}{d\phi^2} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2 \text{ (Say)}$$

Where λ^2 is a separation constant. Thus,

$$\frac{d\beta}{dt} + \lambda^2 \alpha \beta = 0$$

$$\int \frac{d\beta}{dt} = -\lambda^2 \alpha \int dt$$

$$e^{log\beta} = e^{-\lambda^2 \alpha t}$$

$$\beta = C_1\, e^{-\lambda^2 \alpha t} \qquad (2.3)$$

Also,

$$r^2 \sin^2(\theta)\left[\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\sin\theta}\frac{1}{H}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)+\lambda^2\right]=\frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2}=m^2 \text{ (say)}$$

Which gives

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0$$

Whose solution is

$$\Phi(\phi)=C_1 e^{im\phi}+C_2 e^{-im\phi} \qquad (2.4)$$

The other separated equation is

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{1}{r^2\sin\theta}\frac{1}{H}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)+\lambda^2=\frac{m^2}{r^2\sin^2(\theta)}=n(n+1)$$

(or)

$$\frac{r^2}{R}\left(R''+\frac{2}{r}R'\right)+\lambda^2 r^2=\frac{m^2}{\sin^2(\theta)}-\frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)=n(n+1) \text{ (say)}$$

On Re- arrangement, this equation can written as

$$R''+\frac{2}{r}R'+\left\{\lambda^2-\frac{n(n+1)}{r^2}\right\}R=0 \qquad (2.5)$$

And $\frac{-1}{H\sin\theta}\left(\sin\theta\frac{d^2H}{d\theta^2}+\cos\theta\frac{dH}{d\theta}\right)+\frac{m^2}{\sin^2(\theta)}=n(n+1)$

$$\text{(or)}\ \frac{d^2H}{d\theta^2}+\cot\theta\frac{dH}{d\theta}+\left\{n(n+1)-\frac{m^2}{\sin^2(\theta)}\right\}H=0 \qquad (2.6)$$

Let R $=(\lambda r)^{-1/2}\Psi(r)$ then Eq(2.6) becomes

$$=(\lambda r)^{-\frac{1}{2}}\left[\Psi''+\frac{1}{r}\Psi'(r)+\left\{\lambda^2-\frac{(n+1/2)^2}{r^2}\right\}\Psi\right]=0 \qquad \text{since } (\lambda r)\neq 0$$

We have

$$\Psi''(r)+\frac{1}{r}\Psi'(r)+\left\{\lambda^2-\frac{(n+1/2)^2}{r^2}\right\}\Psi(r)=0$$

The above equation is the Bessel's equation of order (n+1/2), whose solution is

$$\Psi(r) = AJ_{n+1/2}(\lambda r)+BY_{n+1/2}(\lambda r)$$

$$R(r)= =(\lambda r)^{-\frac{1}{2}} [AJ_{n+1/2}(\lambda r)+BY_{n+1/2}(\lambda r)] \qquad (2.7)$$

Where Jn and Yn are Bessel's function of first and second kind respectively.

Now equation(2.7) can be put in a more convenient form by introducing a new independent variable

$$\mu=\cos\theta \ \ (\cot\theta = \mu/\sqrt{1-\mu^2}, \frac{dH}{d\theta} = -\sqrt{1-\mu^2}\frac{dH}{d\mu}, \ \frac{d^2H}{d\theta^2} = (1-\mu^2)\frac{d^2H}{d\mu^2} - \mu\frac{dH}{d\mu})$$

Thus(2.6) equation becomes

$$(1-\mu^2)\frac{d^2H}{d\mu^2} - 2\mu\frac{dH}{d\mu}+[n(n+1)-\frac{m^2}{1-\mu^2}]H=0$$

Which is an associated legendra differential equation. Whose solution is

$$H(\theta)=A_1P_n^m(\mu)+A_2Q_n^m(\mu)$$

Where $P_n^m(\mu)$ and $Q_n^m(\mu)$ are legendra function of degree n order m, of first and second kind, respectively.

The physically meaningful general solution of the diffusion equation in spherical geometry is of the form

$$T(r,\theta,\phi,t)=\sum_{\lambda,m,n} A_{\lambda mn}(\lambda r)^{-\frac{1}{2}} J_{n+1/2}(\lambda r)P_n^m(\cos\theta)\, e^{\pm im\phi-\alpha\lambda^2 t}$$

In this general solution, the function $Q_n^m(\mu)$ and $(\lambda r)^{-\frac{1}{2}} Y_{n+1/2}(\lambda r)$ are excluded because these function have poles at $\mu=\pm 1$ and r=0 respectively.

Example:2

Find the temperature in a sphere of radius a. when its surface is kept at 0 temperature and its initial temperature is f(r, θ).

Solution:

Here, the temperature is governed by the 3-D heat equation in spherical polar coordinates independent of therefore, the task is to find the solution of PDE .

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial T}{\partial\theta}) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.8)$$

Subject to

Boundary condition :T(a, θ,t)=0 (2.9)

Initial condition : T(r,θ,0)=f(r,0) (2.10)

The general solution of equation (2.8) with the help of Eq(2.9), can be written as

$$T(r,\theta,t)=\sum_{\lambda,n} A_{\lambda n}(\lambda r)^{-\frac{1}{2}} J_{n+1/2}(\lambda r)P_n(\cos\theta)\, e^{-\alpha\lambda^2 t} \quad (2.11)$$

Applying the boundary condition we get,

$$J_{n+1/2}(\lambda a)=0$$

This equation has infinitely many positive roots. Denoting them by ε_i ,we have

$$T(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{i=1}^{\infty} A_{n_i}(\varepsilon_i r)^{-1/2} J_{n+1/2}(\varepsilon_i r)P_n(\cos\theta)\exp(-\alpha\,\varepsilon_i^{2} r) \quad (2.12)$$

Now applying the initial condition and denote $\cos\theta$ by μ, we get

$$f(r,\cos^{-1}(\mu) = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} A_{n_i}(\varepsilon_i r)^{-1/2} J_{n+1/2}(\varepsilon_i r)P_n(\mu)$$

Multiply both sides by $P_m(\mu)$ and integrating between the limits, -1 to 1,we obtain

$$\int_{-1}^{1} f(r,\cos^{-1}(\mu)P_m(\mu)d\mu = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} A_{n_i}(\varepsilon_i r)^{-1/2} J_{n+1/2}(\varepsilon_i r)\int_{-1}^{1} P_m(\mu)P_n(\mu)d\mu$$

$$= \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} A_{n_i}(\varepsilon_i r)^{-1/2} J_{n+1/2}(\varepsilon_i r)\left(\frac{2}{2n+1}\right)$$

(or)

$$\left(\frac{2}{2n+1}\right)\int_{-1}^{1} f(r,\cos^{-1}(\mu)P_m(\mu)d\mu = \sum_{i=1}^{\infty} A_{n_i}(\varepsilon_i r)^{-1/2} J_{n+1/2}(\varepsilon_i r) \quad \text{for } n=0,1,2....$$

Now, to evaluate the constant A_{n_i}

Multiply both side of the above equation by $r^{3/2} J_{n+1/2}(\varepsilon_i r)$ and integrate with respect to r get

$$\varepsilon_i^{1/2}\left(\frac{2}{2n+1}\right)\int_0^a r^{3/2}\, J_{n+1/2}(\varepsilon_i r)dr\left(\frac{2}{2n+1}\right)\int_{-1}^1 f(r,\cos^{-1}(\mu))P_m(\mu)d\mu = \sum_{i=1}^{\infty} A_{n_i}\int_0^a r\, J_{n+1/2}(\varepsilon_i r)\, J_{n+1/2}(\varepsilon_i r)dr$$

$$= \frac{1}{2}\sum_{i=1}^{\infty} A_{n_i}[\, J_{n+1/2}'(\varepsilon_i r)]^2,\ n=0,1,2\ldots \qquad (2.13)$$

Thus, equations (2.12) and (2.13) together constitutes the solution for the given problem.

Conclusion

The expectation of using variables separable method and obtaining better results, in a very expressive way was achieved.