RADIALLY SYMMETRIC SOLUTIONS OF SYSTEM OF NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract: Symmetry properties of positive solutions of system of elliptic boundary value problems in R^3 are studied. We employ the moving plane method based on maximum principle on unbounded domains to obtain the result on symmetry of solutions of systems of nonlinear elliptic boundary value problems.

IndexTerms :- Maximum principle; Moving plane method; Radial symmetry; System of nonlinear elliptic boundary value problem.

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I. INTRODUCTION

In recent years, a lot of interest has been shown in the study of symmetry properties of solutions of nonlinear elliptic equations, reflecting the symmetry of domain. Linear elliptic equations arise in several models describing various phenomena in the Applied sciences. It is an important goal in mathematical analysis to establish symmetry property of solutions of differential equations both from a theoretical point of view and for the applications .In 1979 Gidas, Ni and Nirenberg [9, 10] introduced the method of moving planes to obtain the symmetry results and monotonicity for positive solutions of nonlinear elliptic equations. Li and Ni[19] proved the symmetry results for the conformal scaler curvature equation

$$\Delta u + K(x)u^{(n+2)/(n-2)} = 0 \text{ in } R^n;\ n \geq 3.$$

Recently, Naito [16] studied the problem of radial symmetry of classical solutions of semilinear elliptic equations

$$\Delta u+V(|x|)\, e^u = 0 \text{ in } R^2,$$

by the moving plane method. Recently Dhaigude and Patil [2] studied the radial symmetry of positive solutions of semilinear elliptic problem in unit ball. Also in [4] authors obtained symmetric results for semilinear elliptic boundary value problems (BVP) $\Delta u + f(|x|\ u; \nabla u) = 0$ in R^n $u(x) \to 0$ as $|x| \to \infty$.

In [5] authors studied the symmetry of solutions of elliptic boundary value problem

$$\Delta u + V(|x|)\, e^u = 0 \text{ in } R^3,$$

using moving plane method. Dhaigude and Patil [3] have proved some symmetry result for the system of nonlinear BVP. Further author [6] proved that solution of a system of semilinear elliptic Neumann boundary value problem must be radially symmetric. Author [7] also proved radial symmetry of positive solutions for elliptic system with gradient term in R^n. Symmetric solutions of nonlinear elliptic Neumann boundary value problems are obtained by author D. P. Patil[8]. It is well known that a classical tool to study symmetry of such solutions is the moving plane method. Which goes back to Alexandrov and Serrin [20]. Since last four decades or so, "method of moving planes" has been numerous applications in studying non linear partial differential equations. It can be used to prove symmetry of solutions. Method of moving planes has been improved and simplified by Berestycki and Nirenberg in [21] with the aid of maximum principle in small domain. After that many other results followed with different operators, different boundary conditions and different geometries.

In this paper the nonlinear system of the following form

$$\Delta u(x) + g_1(|x|)\, e^{u(x)} + g_2(|x|)\, e^{v(x)} = 0 \quad \text{in } R^3 \ (1.1)$$

$$\Delta v(x) + g_3(|x|)\, e^{u(x)} + g_4(|x|)\, e^{v(x)} = 0;$$

$$u(x) \to 0;\ v(x) \to 0 \text{ as } |x| \to \infty \qquad (1.2)$$

where $g_1(x); g_2(x); g_3(x); g_4(x)$ are locally Holder continuous in $(0;\infty)$, satisfying $u \in L^\infty(R^3)$ and $v \in L^\infty(R^3)$ is studied. Naito [13] studied dirichlet nonlinear BVP

$$\Delta u(x) + f(|x|; u(x)) = 0 \text{ in } R^2$$

$$u(x) \to 0 \text{ as } |x| \to \infty$$

Problems of this kind arise in geometry and various branches of physics; see Chanillo and Kiessling [14]. Chen and Li [17] explained and used moving plane method in proving symmetry of solutions of BVP. Chen and Zhu [18] proved radial symmetry of solutions of polyharmonic Dirichlet boundary value problem

$$(-\Delta)^m u = f(u)$$

in unit ball by using the corresponding integral equation and moving plane method in integral form. Recently, Hui and Kim [12] proved the symmetry of solution of the equation

$$\Delta v + \alpha e^v + \beta\, x{:}\, \nabla e^v = 0 \text{ in } R^n.$$

We organize the paper as follows: Section 2 is devoted to some preliminary results which will be used in next section. The symmetric result and useful lemmas are proved in the last section.

2. PRELIMINARY RESULTS:

In this section, first we state some lemmas and theorem which are useful to prove our main result.

Lemma 2.1: *Suppose that* Ω satisfies *the interior sphere condition at* $x_0 \in \partial\Omega$. *Let L be strictly elliptic with* $c \leq 0$. *If* $u \in C^2(\Omega) \cap C(\Omega)$ *satisfies* $L(u) \geq 0$ *and* $\max \Omega\, u(x) = u(x_0)$ *then either* $u = u(x_0)$ *on* Ω *or* $\liminf_{t\to 0} (u(x) - u(x_0 + tv)/t > 0$, for every *direction v, pointing into an interior sphere. If* $u \in C^1$ *subset of* $\Omega\, U\{0\}$ *then* $\partial u/\partial v\,(x_0) < 0$.

Lemma 2.2 *[16] Let* Ω *be unbounded domain in* R^n, *and let L denote the uniformly elliptic differential operator of the form* $Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u$
where a_{ij} ; b_i; $c \in L^\infty(\Omega)$. *Suppose that* $u \neq 0$ *satisfies* $L(u) \leq 0$ *in* Ω *and* $u \geq 0$ *on* $\partial\Omega$: *Suppose furthermore that there exist a function w such that* $w > 0$ *on* $\Omega\, U\, \partial\Omega$ *and* $L(w) \leq 0$ *in* Ω. *If* $u(x)/w(x) \to 0$ $|x| \to \infty$; $x \in \Omega$
then

$$u > 0 \text{ in } \Omega:$$

Theorem 2.1 *[11] Let* $u(x)$ *satisfies differential inequality* $L(u) \geq 0$ *in a domain* Ω *where L is uniformly elliptic. If there exist a function w(x) such that,*

$$w(x) > 0 \text{ on } \Omega\, U\, \partial\Omega.\ L(w) \leq 0 \text{ in } \Omega.$$

Then $u(x) / w(x)$ *cannot attain a non negative maximum at a point p on* $\partial\Omega$, *which lies on the boundary of a ball in* Ω *and if u/w is not constant then*

$$,\ \partial/\partial v\,(u/w) > 0$$

at P where $\partial/\partial v$ *is any outward directional derivative.*

Let w be harmonic function in R^3 and $w(x) = o(|x|)$ as $|x| \to \infty$. Then w must be constant.

Lemma 2.3 *[11] Let f be bounded and locally Holder continuous in* Ω *and let w be the Newtonian potential of f. Then* $w \in C^2(\Omega)$, $\Delta w = f$ *in* Ω *and for any*

$$x \in \Omega\ D_{ij}\, w(x) = \int_{\Omega_0} D_{ij}\, \Gamma(x-y)(f(y) - f(x))dy - \int_{\partial\Omega_0} D_i\Gamma(x-y)v_j(y)d_{sy}$$

here Ω_0 *is any domain containing* Ω *for which the divergence theorem holds and is extended to vanish outside* Ω.

Lemma 2.4 *[5] If* $w(x) = 1/3\omega_3 \int_{R3} (1/|x-y| - |y|)\, f(y)dy$.
Let $f \in L^\infty(R^3) \cap L^1(R^3)$. *Then*

$$\lim_{|x|\to\infty} w(x)/\log|x| = 1/3\omega_3 \int_{R^3} f(y)\, dy.$$

3. MAIN RESULTS:

Before going to prove main result we prove some lemmas.

Lemma 3.1 *Let* $(u; v)$ *be solution of* (1.1) *where* $g_1(x)$; $g_2(x)$; $g_3(x)$; $g_4(x)$ *are locally Holder continuous in* $(0;1)$, *satisfying* $u+ \in L^\infty(R^3)$ *and* $v+ \in L^\infty(R^3)$ *and* $0 < 1/3\omega_3 \int_{R3} g_1(|x|)e^{u(x)} + g_2(|x|)e^{v(x)}dx = \beta_1 < \infty$ (3.1)

and $0 < 1/3\omega_3 \int_{R3} g_3(|x|)e^{u(x)} + g_4(|x|)e^{v(x)}dx = \beta_2 < \infty$ (3.2)

Then $\lim_{|x|\to\infty} u(x)/\log|x| = \lim_{|x|\to\infty} u(x^\lambda)/\log|x| = -\beta_1$ (3.3) *and* $\lim_{|x|\to\infty} v(x) / \log |x| = \lim_{|x|\to\infty} v(x^{\lambda}) / \log |x| = -\beta_2$ (3.4)

Proof: Define the functions $w_1(x)$ and $w_2(x)$ as,

$$w_1(x) = 1/3\omega_3 \int_{R3} (1/|x-y| - |y|) g_1(|y|)e^{u(y)} + g_2(|y|)e^{v(y)}dy \text{ in } R^3 \quad (3.5)$$

$$w_2(x) = 1/3\omega_3 \int_{R3} (1/|x-y| - |y|) g_3(|y|)e^{u(y)} + g_4(|y|)e^{v(y)}dy \text{ in } R^3 \quad (3.6)$$

As $x_1 \neq x_2$ we have $w_1(x_1) \neq w_1(x_2)$ and $w_2(x_1) \neq w_2(x_2)$, and so $w_1(x)$ and $w_2(x)$ are well defined, and

$$\Delta w_1 = g_1(|y|)e^{u(y)} + g_2(|y|)e^{v(y)} \quad (3.7)$$

and

$$\Delta w_2 = g_3(|y|)e^{u(y)} + g_4(|y|)e^{v(y)} \quad (3.8)$$

From equations (3.3), (3.4) and $u+; v+ \epsilon L^{\infty}(R^3)$ we have $g_1(|x|)e^{u(x)} + g_2(|x|)e^{v(x)} \epsilon L^{\infty}(R^3) \cap L^1(R^3)$ and $g_3(|x|)e^{u(x)} + g_4(|x|)e^{v(x)} \epsilon L^{\infty}(R^3) \cap L^1(R^3)$. By lemma 2.1 we have

$$\lim_{|x|\to\infty} w_1(x)/ \log |x| = 1/3\omega_3 \int_{R3} g_1(|y|)e^{u(y)} + g_2(|y|)e^{v(y)} = \beta_1 < \infty \quad (3.9)$$

$$\lim_{|x|\to\infty} w_2(x)/ \log |x| = 1/3\omega_3 \int_{R3} g_3(|y|)e^{u(y)} + g_4(|y|)e^{v(y)} = \beta_2 < \infty \quad (3.10)$$

Therefore we have $\lim_{|x|\to\infty} w_1(x)/ \log |x| \;\; \beta_1 < \infty$ and $\lim_{|x|\to\infty} w_2(x)/ \log |x| = \beta_1 < \infty$ (3.11)

Before going to next lemma we shall define,

$$w_{1;\lambda\_}(x) = u(x) - u(x^{\lambda})$$
$$w_{2;\lambda\_}(x) = v(x) - v(x^{\lambda})$$

Lemma 3.2 *Let* $\lambda > 0$ *then* $w_{1;\lambda\_}(x)$ *and* $w_{2;\lambda\_}(x)$*satisfies*

$$\Delta w_{1;\lambda\_}(x) + C_1;_{\lambda}(x) w_{1;\lambda\_}(x) \leq 0$$
$$\Delta w_{2;\lambda\_}(x) + C_2;_{\lambda}(x) w_{2;\lambda\_}(x) \leq 0$$

where

$$C_1;_{\lambda}(x) = O(|x|)^{-\delta 1}) \text{ and } C_2;_{\lambda}(x) = O(|x|)^{-\delta 2}) \text{ as } |x| \to\infty$$

for some constants $\delta_1; \delta_2 > 3$.

Proof: Since $g_i(|x|); i = 1; 2; 3; 4:$ is nonincreasing in $|x| = r$ and $|x^{\lambda}| > |x|$ for $x \epsilon \sum_{\lambda}$ and $\lambda > 0$ we know that $u(x^{\lambda})$ satisfies the same equations that $u(x)$ does.

$$\Delta u(x_{\lambda}) + g_1(|x^{\lambda}|)e^{u(x\lambda)} + g_2(|x^{\lambda})e^{v(x\lambda)} = 0 \quad (3.12)$$

and

$$\Delta v(x_{\lambda}) + g_3(|x^{\lambda}|)e^{u(x\lambda)} + g_4(|x^{\lambda})e^{v(x\lambda)} = 0 \quad (3.13)$$

Subtracting equation (3.12) from equation (1.1) we get,

$$0 = [\Delta u(x) + g_1(|x|)e^{u(x)} + g_2(|x|)e^{v(x)}] - [\Delta u(x^{\lambda}) + g_1(|x^{\lambda}|)e^{u(x\lambda)} + g_2(|x^{\lambda})e^{v(x\lambda)}]$$
$$= \Delta u(x) - \Delta u(x^{\lambda}) + [g_1(|x|)e^{u(x)} + g_2(|x|)e^{v(x)} - g_1(|x^{\lambda}|)e^{u(x\lambda)} - g_2(|x^{\lambda})e^{v(x\lambda)}]$$
$$\geq \Delta w_{1;\lambda\_}(x) + [g_1(|x|)e^{u(x)} - g_1(|x|)e^{u(x\lambda)}] + [g_2(|x|)e^{v(x)} - g_2(|x^{\lambda})e^{v(x\lambda)}]$$
$$= \Delta w_{1;\lambda\_}(x) + g_1(|x| [e^{u(x)} - e^{u(x\lambda)}] + g_2(|x|) [e^{v(x)} - e^{v(x\lambda)}]$$
$$= \Delta w_{1;\lambda\_}(x) + \{[g_1(|x| [e^{u(x)} - e^{u(x\lambda)}] + g_2(|x|) [e^{v(x)} - e^{v(x\lambda)}]]/ (u(x) - u(x^{\lambda})\} (u(x) - u(x^{\lambda})$$
$$= \Delta w_{1;\lambda}(x) + C_{1;\lambda\_}(x)(w_{1;\lambda\_}(x))$$

Where

$$C_{1;\lambda\_}(x) = [g_1(|x| [e^{u(x)} - e^{u(x\lambda)}] + g_2(|x|) [e^{v(x)} - e^{v(x\lambda)}]]/ (u(x) - u(x^{\lambda})$$

Similarly,

$$0 \geq \Delta w_{2;\lambda}(x) + C_{2;\lambda\_}(x)(w_{2;\lambda\_}(x))$$

where

$$C_{2;\lambda\_}(x) = [g_3(|x| [e^{u(x)} - e^{u(x\lambda)}] + g_4(|x|) [e^{v(x)} - e^{v(x\lambda)}]]/ (v(x) - v(x^{\lambda})$$

Take $\epsilon > 0$ so small that $\alpha_i + \beta_j - \epsilon > 3$ for $i = 1; 2; 3; 4$. $j = 1; 2$. Then we have

$$u(x)/ \log |x| \leq -(\beta_1 - \epsilon),\ u(x) \leq -(\beta_1 - \epsilon) \log |x|;\ u(x^{\lambda})/ \log |x| \leq -(\beta_1 - \epsilon),$$

$$v(x)/ \log |x| \leq -(\beta_2 - \epsilon),\ v(x) \leq -(\beta_2 - \epsilon) \log |x|,\ v(x^{\lambda})/ \log |x| \leq -(\beta_2 - \epsilon)$$

Thus we have $u(x) \leq -(\beta_1 - \epsilon) \log |x|$ and $u(x^{\lambda}) \leq -(\beta_1 - \epsilon) \log |x|$ for all large $|x|$. Also $v(x) \leq -(\beta_2 - \epsilon) \log |x|$ and $v(x^{\lambda}) \leq -(\beta_2 - \epsilon) \log |x|$ for all large $|x|$. From $r^{\alpha i}g_i(r) < \infty$, for some $\alpha_i > 0$ Thus we can conclude that $C_{1;\lambda\_}(x) = o(|x|^{-(\alpha 1+\beta 1-\epsilon)})$ and $C_{1;\lambda\_}(x) = o(|x|^{-(\alpha 2+\beta 2-\epsilon)})$ as $|x|\rightarrow\infty$. This implies the result with $\alpha_1+\beta_1-\epsilon = \delta_1$ and $\alpha_2+\beta_2-\epsilon = \delta_2$. By virtue of lemma 3.2 we can take $R_0 > \epsilon$ so large that,

$$1/ |x| + |x|^2 \log (1/ |x|) \geq C_{1;\lambda\_}(x) \tag{3.14}$$

and

$$1/ |x| + |x|^2 \log (1/ |x|) \geq C_{2;\lambda\_}(x) \tag{3.15}$$

Lemma 3.3 *Let* $\lambda > 0$ *If* $w_{1;\lambda}(x) > 0$ *and* $w_{2;\lambda}(x) > 0$ *on* $\sum_{\lambda} \setminus \boldsymbol{B_{R0}}$ *then* $\lambda \epsilon \Lambda$.

Proof: By lemma 3.2 and assumptions we have,

$$\Delta w_{1;\lambda}(x)+ C_{1;\lambda\_}(x)\, w_{1;\lambda}(x) \leq 0 \text{ in } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}}$$
$$w_{1;\lambda}(x) \text{ on } \partial(\textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}})$$

and

$$\Delta w_{2;\lambda}(x)+ C_{2;\lambda\_}(x)\, w_{2;\lambda}(x) \leq 0 \text{ in } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}}$$

$$W_{2;\lambda}(x) \text{ on } \partial(\textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}})$$

Let $w(x) = 1/ |x| + \log(1/ |x|)$: Then

$$\Delta w +1/ |x|^2 = 0$$

$$\Delta w +1/ \{|x|^2(1/ |x| + \log (1/ |x|))\}w= 0 \quad \text{in } R^3 \setminus \{0\}$$

By equation (3.14) and (3.15) we have

$$\Delta w + C_{1;\lambda\_}(x)\, w \leq 0 \text{ in } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}} \text{ and } w > 0 \text{ on closure of } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}}$$

and

$$\Delta w + C_{2;\lambda\_}(x)\, w \leq 0 \text{ in } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}} \text{ and } w > 0 \text{ on closure of } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}}$$

By lemma 3.1 we have

$$w_{1;\lambda}(x)/ w(x)= [(u(x) - u(x^{\lambda}))/\log(|x|)].[\log(|x|) /w(x)] \rightarrow 0 \text{ as } |x|\rightarrow\infty.$$

and

$$w_{2;\lambda}(x)/ w(x)= [(v(x) - v(x^{\lambda}))/\log(|x|)].[\log(|x|) /w(x)] \rightarrow 0 \text{ as } |x|\rightarrow\infty.$$

By lemma 2.2, we have

$$w_{1;\lambda}(x) > 0 \text{ in } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}}$$

and

$$w_{2;\lambda}(x) > 0 \text{ in } \textstyle\sum_{\lambda} \setminus \boldsymbol{B_{R0}}$$

Then by the assumptions,

$$w_{1;\lambda}(x) > 0 \text{ in } \textstyle\sum_{\lambda}$$

and

$$w_{2;\lambda}(x) > 0 \text{ in } \textstyle\sum_{\lambda}$$

This implies that $\lambda \epsilon \Lambda$.

Lemma 3.4 *Let* $\lambda \epsilon \Lambda$ *then* $\partial u/ \partial x_1 < 0$.

Proof: By lemma 3.2 we have,

$$\Delta w_{1;\lambda}(x)+ C_{1;\lambda\_}(x)\, w_{1;\lambda}(x) \leq 0 \text{ in } \textstyle\sum_{\lambda}$$

$$\Delta w_{2;\lambda}(x)+ C_{2;\lambda\_}(x)\, w_{2;\lambda}(x) \leq 0 \text{ in } \textstyle\sum_{\lambda}$$

And

$$\Delta w_{1;\lambda}(x) > 0,\ \Delta w_{2;\lambda}(x) > 0 \text{ in } \sum_\lambda.$$

Since $\Delta w_{1;\lambda}(x)= 0$, $\Delta w_{2;\lambda}(x)= 0$ on T_λ, we obtain $\partial w_{1;\lambda} / \partial x_1 < 0$ and $\partial w_{2;\lambda} / \partial x_1 < 0$ on T_λ by the Hopf boundary lemma. Therefore we have,

$$\partial u / \partial x_1 = (1/2)\, \partial w_{1;\lambda} / \partial x_1 \ \text{ on } T_\lambda$$
$$\partial v / \partial x_1 = (1/2)\, \partial w_{1;\lambda} / \partial x_1 \ \text{ on } T_\lambda$$

Now we are well equipped to prove our main result about symmetry.

Theorem 3.1 *Assume that g_1, g_2, g_3, g_4 satisfies*

$$\limsup_{|x|\to\infty} r^{\alpha i} g_i(r) < 1 \text{ for some } \alpha_i > 0;\ i = 1;\ 2;\ 3;\ 4. \qquad (3.16)$$

Let $(u; v)$ be the solution of the system (1.1)satisfying $u+ \epsilon L^\infty(R^3)$; $v+ \epsilon L^\infty(R^3)$ where

$$u+ = \max\{0; u\} ;\ v+ = \max\{0; v\}$$

and

$$0 < 1/ 3\omega_3 \int_{R3} g_1(|x|)e^{u(x)} + g_2(|x|)e^{v(x)}dx = \beta_1 < \infty$$

and

$$0<1/3\omega_3\int_{R3} g_3(|x|)e^{u(x)} +g_4(|x|)e^{v(x)}dx=\beta_2 <\infty$$

If $\alpha i + \beta j > 3$, then (u ,v) must be radially symmetric.

Proof: From lemma 3.1 we have,

$$\lim_{|x|\to\infty} u(x)/ \log(|x|) = -\beta_1 ;\ \lim_{|x|\to\infty} v(x)/ \log(|x|) = -\beta_2$$

$$\lim_{|x|\to\infty} u(x)= - \infty \text{ and } \lim_{|x|\to\infty} v(x)= - \infty$$

then there exist $R_1 > R_0$ and $R_2 > R_0$ such that

$$\max\{u(x) : |x| \geq R_1\} \leq \min\{u(x) : |x| \geq R_0\}$$

and

$$\max v(x) : \{v(x) : |x| \geq R_2\}\leq \min\{v(x) : |x| \geq R_0\}$$

Let $Rm = \max \{R_1;R_2\}$, therefore we have

$$\mathrm{Max}\{u(x) : |x| \geq R_m\} \leq \min\{u(x) : |x| \geq R_0\}$$

and

$$\mathrm{Max}\{v(x) : |x| \geq R_m\} \leq \min\{u(x) : |x| \geq R_0\}$$

where R_0 is constant such that

$$1/ |x| + |x|^2 \log (1/ |x|) \geq C_{1;\lambda\_}(x)$$

and

$$1/ |x| + |x|^2 \log (1/ |x|) \geq C_{2;\lambda\_}(x) \quad \text{for some } |x| \geq R_0.$$

We shall prove the theorem in three steps.

Step-I: To prove $[Rm;\infty)$ *subset* Λ.
Let

$$\lambda\, \epsilon[Rm;\infty) \text{ , therefore } \lambda\geq Rm:$$

We know that $\boldsymbol{BR_0}$ subset $\sum_\lambda$. Also we have

$$w_{1;\lambda\_}(x) > 0,\ w_{2,\lambda}(x) > 0 \text{ in } BR_0$$

Thus $\lambda\epsilon\Lambda$. Therefore

$$[Rm;\infty) \text{ is } subset\ \Lambda.$$

Step-II: Let $\lambda_0 \,\epsilon\, \Lambda$, then we have to prove that there exist $\epsilon > 0$ such that $(\lambda_0 - \epsilon; \lambda_0]$ *subset* Λ. We shall prove this step by the method of contradiction. Suppose that there exist an increasing sequence $\{\lambda_i\}$; $i = 1; 2; 3; :::$, such that $\lambda_i\, \epsilon\Lambda$ does not hold and $\lambda_i \to \lambda_0$ as $i \to \infty$. By lemma 3.2 we have a sequence $\{x_i\}, i = 1; 2; 3; :::$ such that

$$xi \epsilon \sum_{\lambda i} \cap \boldsymbol{BR_0} \text{ and } w_{1;\lambda i\_}(x_i) \le 0,\ w_{1;\lambda i\_}(x_i) \le 0.$$

Then there exist a subsequence which we call again sequence $\{xi\}$ converges to some point $x_0 \epsilon \sum_{\lambda i} \cap \boldsymbol{BR_0}$ **.** Then

$$w_{1;\lambda 0\_}(x_0) \le 0 \text{ and } w_{2;\lambda 0\_}(x_0) \le 0 \ .$$

Since $w_{1;\lambda 0\_} > 0$ and $w_{2;\lambda 0\_} > 0$ *;* in $\sum_{\lambda 0}$ .We have $x_0\ \epsilon T_{\lambda 0}$. By mean value theorem we see that there exist a point y_i satisfying

$$\partial u / \partial x_1(y_i) \ge 0 \text{ and } \partial v / \partial x_1(y_i) \ge 0$$

on the straight segment joining x_i *to* $(x_i)^{\lambda i}$ for each $i = 1;\ 2;\ 3;\ ::::$ Since $y_i \to x_0$ *as* $i \to \infty;$ we have

$$\partial u / \partial x_1(x_0) \ge 0 \text{ and } \partial v / \partial x_1(x_0) \ge 0.$$

On the other hand, since $x_0\ \epsilon T_{\lambda 0}$, we have

$$\partial u / \partial x_1(x_0) < 0 \text{ and } \partial v / \partial x_1(x_0) < 0.$$

This is a contradiction; hence step (II) is proved.

Step-III: Consider the following two statements (A) and (B),

(A) $u(x) = u(x^{\lambda 1});\ v(x) = v(x^{\lambda 1})$ for some $\lambda_1 > 0$
and

$$\partial u / \partial x_1(x_0) < 0 \ ,\ \partial v / \partial x_1(x_0) < 0 \ on\ T_\lambda \quad \text{for some } \lambda > \lambda_1.$$

(B) $u(x) = u(u^0);\ v(x) = v(x^0)$ for some $\lambda_0 > 0$
and

$$\partial u / \partial x_1(x_0) < 0 \ ,\ \partial v / \partial x_1(x_0) < 0 \ on\ T_\lambda \quad \text{for some } \lambda > 0.$$

We have to prove that either (A) or (B) holds.
If $g_1;\ g_2;\ g_3;\ g_4;$ are not constant then statement (B) must holds.
Let $\lambda_1 = \inf\{\lambda > 0 : [\lambda;\infty) \text{subset } \Lambda\}$.
We distinguish the proof in two cases: (i)$\lambda_1 > 0$ and (ii) $\lambda_1 = 0$*:*
Case (i): The case where $\lambda_1 > 0$*:* Let

$$w_{1;\lambda 1\_}(x) = u(x) - u(x^{\lambda 1}) \text{ and } w_{2;\lambda 1\_}(x) = v(x) - v(x^{\lambda 1}).$$

Since u and v are continuous we have

$$w_{1;\lambda 1\_}(x) \ge 0;\ w_{1;\lambda 1\_}(x) \ge 0 \text{ ,in} \le \sum_{\lambda 1} .$$

From lemma 3.2, we have

$$\Delta\ w_{1;\lambda 1}(x) + C_{1;\lambda 1\_}(x)\ w_{1;\lambda 1}(x) \le 0$$

and

$$\Delta\ w_{2;\lambda 1}(x) + C_{2;\lambda 1\_}(x)\ w_{2;\lambda 1}(x) \le 0$$

Hence, by strong maximum principle we have that either

$$w_{1;\lambda 1} > 0;\ w_{2;\lambda 1} > 0; \text{ in } \sum_{\lambda 1}$$

or

$$w_{1;\lambda 1} = 0;\ w_{2;\lambda 1} = 0; \text{ in } \sum_{\lambda 1}.$$

Assume that

$$w_{1;\lambda 1} > 0;\ w_{2;\lambda 1} > 0 \ \text{ in } \sum_{\lambda 1} \text{ then } \lambda_1\ \epsilon\ \lambda.$$

From step (II) there exist $\epsilon > 0$ such that $(\lambda_1 - \epsilon;\ \ \lambda_1]$ .This contradicts definition of λ_1*:* so

$$w_{1;\lambda 1} = 0;\ w_{2;\lambda 1} = 0;$$

Since $(\lambda_1;\infty)$ *subset of* Λ , we have

$$\partial u/\partial x_1(x_0) < 0 \, , \, \partial v/\partial x_1(x_0) < 0 \text{ on } T_\lambda$$

Thus we get statement (A).

Case (ii): The case where $\lambda_1 = 0$. From continuity of $u; v$ we have

$$w_{1;\lambda 0}(x) \geq 0 \, , \, w_{2;\lambda 0}(x) \geq 0 \text{ in } \sum_0$$

i.e.

$$u(x) \geq u(x^0) \text{ in } v(x) \geq v(x^0) \text{ in } \sum_0 \, ,$$

by lemma 3.4, we have,

$$\partial u/\partial x_1 < 0 \, , \, \partial v/\partial x_1 < 0 \text{ on } T_\lambda \text{ for } \lambda > 0$$

Assume that $g_1; g_2; g_3; g_4$ are not constant, in this case we have to prove that (B) must hold. Assume to the contrary that (A) holds. From (1.1) we have,

$$g_i(|x|) = g_i(|x^\lambda|) \text{ for } x \in \sum_\lambda \text{ and; } i = 1; 2; 3; 4.$$

Since $g_i(r)$ are non increasing, we have g_i are constant. This contradicts to the assumption. Thus (B) holds.
If (B) occurs in step (III) then we can repeat all the three steps for negative X_1- direction about plane

$$x_1 = \lambda_1 < 0$$

or

$$u(x) \leq u(x^0) \; ; \; v(x) \leq v(x^0) \text{ in } \sum_0 \qquad (3.17)$$

If (3.17) occurs then

$$u(x) = u(x^0) \text{ and } v(x) = v(x^0) \text{ in } \sum_0 :$$

Therefore, u and v must be radially symmetric in X_1-direction about some plane and strictly decreasing away from the plane. Since equation (1.1) is invariant under rotation we may take any direction as X_1-direction and conclude that $(u; v)$ is symmetric in every direction about some plane. Therefore, $(u; v)$ is radially symmetric and decreasing the case where g_i are not constant then (B) and (3.17) holds. Then $(u; v)$ is radially symmetric about origin and

$$u_r < 0 \, , \, v_r < 0 \text{ for } |x| = r > 0 :$$

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