Optimal and Uniform Numerical Methods for Singularly Perturbed Reaction Diffusion Problem

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Abstract: A new numerical method for a singularly perturbed boundary value problem using finite difference method is presented in this paper. The specialty of this problem is, it is a problem with boundary layer at both ends of the domain. The method is unconditionally stable, uniform and optimal with respect to the parameter (Singularly perturbation parameter) in the problem. This finite difference method is computationally faster and takes less storage space in modern digital computer. Experimental results are presented to view the applicability of the method with the help of real time problems, using MAT-LAB.

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I. INTRODUCTION

Singular perturbation problems (SPP) occurring in aircraft trajectory guidance, satellite orbit, control system electromagnetic wave propagation, semi-conductor device, fluid dynamics, etc [1-19]. The traditional standard numerical methods will not solve the SPPs due to instability of the numerical solution. And so, explicit exponential fitted schemes have been designed based on fitted operator methods.[1,3-18] To view the initial/ boundary/ interior layers computational methods have been designed using uniform and variable meshes[7-11, 13-15]. In particular, in [7], a two point boundary value problem have been solved using a computational method, in which at the terminal point the solution of the SPP is approximated by the solution of the reduced problem. The region of domain is partitioned into the smooth and transient region. Both the regions are solved by a single exponential fitted operator method with one mesh in the smooth region and another mesh in the transient region. In the transient region an iterative procedure is applied. After the introduction of Shishkin fitted mesh, lot of changes[1,19] in the field of SPPs. Few draw backs are there in Shishkin fitted mesh methods and in fitted operator method, in the sense that, a method designed for a linear SPP cannot be directly extended to non-linear. Similarly cannot be extended from one-space dimension to higher dimensions [1, 19]. In [1], a direction is given to select either fitted operator or fitted mesh methods for a SPP with respect to the real time situation. Both the fitted operator and fitted mesh methods have to be further developed [1].

In [3], using fitted operator, explicit exponentially fitted operator schemes have been designed for linear and non-linear ordinary and partial SPPs. In [4] fitted operator higher order(two) explicit, uniform and optimal methods for first order linear SPPs are designed. In [5], fitted operator method of order one is designed for nonlinear SPP. In [6], using fitted operator method and shooting method a computational procedure is given for second order SPPs with mixed boundary conditions and with left boundary layer. In [7], using fitted operator method and boundary value technique using two different meshes, one mesh for smooth and another mesh for left boundary layer a computational procedure is given. In [8], using fitted operator method a chemical reactor problem is solved. In [9], using fitted operator method and boundary value technique using two different meshes, one mesh(h_1) for smooth and another mesh but same mesh(h_2). for both left and right boundary layers a computational procedure is given., In [10], using fitted operator method one-space dimensional heat equation is solved. In [11], using fitted operator method and initial value technique a computational procedure is given. In [12], using fitted operator method and shooting method a computational procedure is given for SPPs with Dirichlet's conditions with left boundary layer. In [13], a fitted operator method is presented for a non-linear SPP with initial layer. In [14], using fitted operator method and boundary value technique a computational procedure is given as in [9], but with a change in evaluation of solution at terminal points. In [15], using fitted operator method and boundary value technique a computational procedure is given for linear first order SPPs. In [16], using fitted operator method an uniform and optimal method is designed for non-linear SPPs. In [17], a finite difference scheme is presented for non-liner problems. In [18], using fitted operator method an uniform and optimal scheme is given for first order linear SPPs. In [19], a stable numerical method is designed for ball bearing problems using fitted operator methods in [3]. In [3, 20], a full literature survey is given and the 30 years of war in designing numerical methods for SPPs is narrated. In [2], a fitted operator fourth order Numerov method is given for a SPP with multi scale behavior. Some numerical results with absolute error for three test problems are provided. The error estimations are not provided for the continuous problem and numerical convergence is not provided. The graphical performance to view is also not provided.

In this paper, a new uniformly convergent numerical method for a SPP with multi scale nature is designed using fitted operator central difference method and to view twin boundary layers fitted mesh method is applied.

Mathematical modeling of an aircraft optimal control guidance is given in section 2. In Section 3 a domain decomposition method is presented. In section 4, a fitted operator method is presented for the SPP (1a,b). Fitted mesh method for (1a, b) is given in section 5. An algorithm is given in section 6. Final section 7 gives the experimental results using modern digital computer.

Throughout this paper, $\rho = h/\sqrt{\varepsilon}$ and C will be used to denote a generic constant independent of i, h and ε. Error stands for absolute error.

II. Mathematical Modeling

The mathematical modeling for the problem in the study of aircraft optimal control guidance is given by,

$$Lu(t) \equiv -\varepsilon u''(t) + b(t)\,u(t) = f(t),\ 0 < t < 1, \qquad (2.1)$$

$$B_0 u(0) = u(0) = \phi_1,\ B_1 u(1) \equiv u(1) = \phi_2, \qquad (2.2)$$

Where $1 >> \varepsilon > 0$ is a small parameter, ϕ_1 and ϕ_2 are constants, b and f are smooth functions satisfying $b(t) \geq \beta > 0$ for all t Є [0, 1]. The operator L admits maximum principle which is stated in the following theorem [3]

Theorem 2.1. Suppose v is a smooth function satisfying $B_0 v(0) \geq 0$, $B_1 v(1) \geq 0$ and $Lv(t) \geq 0$ for all t in [0,1]. Then, $v(t) \geq 0$ or all t in [0,1].

Proof: Refer[3].

The stability result is given in the following theorem.[3]

Theorem.2.2. Let L be thre operator in (2.1) and v be any smooth function then for all t in [0,1],

$|v(t)| \leq C\,(|v(0)| + |v(1)| + \sup |Lu(s)|)$, s in [0,1] where C is independent of ε.

Proof. Refer[3].

Using Theorem.2.1 we can show that (2.1)-(2.2) has a unique solution and this solution has a boundary layer at each end points. Using Theoorem 2.2, the solution of (2.1)-(2.2) is stable. The reduced problem in this sense is

$$b(t)\, u_0(t) = f(t),\ 0 < t < 1, \qquad (2.3)$$

and we see that, in general, $u_0(t)$ will not satisfy the boundary conditions. As an asymptotic expansion of order zero , we propose[2,3]

$$U(t) = u_0(t) + v_0(\tau) + w_0(\eta) + O(\sqrt{\varepsilon}) \qquad (2.4)$$

where $u_0(t)$ satisties (2.3) and the boundary layer functions $v_0(\tau)$ and $w_0(\eta)$ satisfy the following differential equations :

$$-[d^2/d\tau^2]v_0(\tau) + b(0)v_0(\tau) = 0, \tau \in (0,\omega) \qquad (2.5)$$

$$-[d^2/d\eta]w_0(\eta) + b(1)w_0(\eta) = 0, \eta \in (0,\omega), \qquad (2.6)$$

$$v_0(\tau=0) + w_0(\eta = 1/\sqrt{\varepsilon}) = \phi_1 - u_0(0), \qquad (2.7)$$

$$v_0(\tau = 1/\sqrt{\varepsilon}) + w_0(\eta=0) = \phi_2 - u_0(1), \qquad (2.8)$$

$$v_0(\tau=\omega) + w_0(\eta=\omega) = 0 \qquad (2.9)$$

where $\tau=t/\sqrt{\varepsilon}$ and $\eta=(1-t)/\sqrt{\varepsilon}$. The above equation is obtained by taking Taylors series expansion of b(t) about t=0 and t=1 making change of variables t to τ and t to η and then equating powers of ε. The error estimation between solution of (2.1)-(2.2) and asymptotic expansion for the solution of u(t) is given in the following theorem.

Theorem 2.3. If u is the solution of (2.1)-(2.2) and U is the solution given in (2.4) then for sufficiently smooth b and f

$$|u(t)-U(t)| \le C\sqrt{\varepsilon}. \qquad (2.10)$$

Where C is independent of ε.

Proof. Refer[3].

III. DOMAIN DECOMPOSITION

Decompose the domain [0,1] of the original problem into two equal subdomains as [0,1]=[0,1/2]U[1/2,1] and define the original problem (2.1)-(2.2) into two problems as follows:

$$Lv(t) \equiv -\varepsilon v''(t) + b(t)\, v(t) = f(t), \quad 0< t <1/2, \qquad (3.1)$$

$$B_0 v(0) \equiv v(0) = \phi_1, \quad B_1 v(1/2) \equiv v(1/2) = u_0(1/2) \qquad (3.2)$$

And

$$Lw(t) \equiv -\varepsilon w''(t) + b(t)\, w(t) \;\; f(t), \; 1/2< t<1, \qquad (3.3)$$

$$B_0 w(1/2) \equiv w(1/2) = u_0(1/2), \quad B_1 w(1) \equiv w(1) = \phi_2 \qquad (3.4)$$

Where $u_0(t)$ is as defined in (2.3). Now the solution of the problem (2.1)-(2.2) is defined as

$$u(t) = \begin{cases} v(t) & 0 \le t \le \frac{1}{2} \\ w(t) & 1/2 \le t \le 1 \end{cases}$$

(3.3)-(3.4). respectively are derived using maximum principle.

Theorm 3.1. If u is the solution of (2.1)-(2.2) and v is the solution of (3.1)-(3.2) then for sufficiently smooth b and f

$$|u(t) - v(t)| \le C\sqrt{\varepsilon} \qquad (3.5)$$

Where C is independent of ε.

Proof: For t=0, $u(0) - v(0) = \phi_1 - \phi_1 = 0$,

For $0< t <1/2$, $L[u(t) - v(t)] = Lu(t) - Lv(t)$

$= f(t)-f(t)=0.$

For t=1/2, $u(1/2) - v(1/2) = u(1/2) - u_0(1/2)$

$= v_0(1/2) + w_0(1-1/2\varepsilon) + O(\sqrt{\varepsilon})$

$= O(\sqrt{\varepsilon})$

since $v_0(\tau) = O(\sqrt{\varepsilon})$ and $w_0(\eta) = O(\sqrt{\varepsilon})$.

Using maximum principle,

$$|u(t) - v(t)| \le C\,(|u(0) - v(0)| + |u(1/2) - v(1/2)| + \sup |L\,u(s) - v(s)|) \text{ for s in } [0,1/2]$$

$$\le C\sqrt{\varepsilon}$$

Hence the desired result.

Theorm 3.2. If u is the solution of (2.1)-(2.2) and w is the solution of (3.3)-(3.4) then for sufficiently smooth b and f

$$|u(t) - w(t)| \le C\sqrt{\varepsilon}. \qquad (3.6)$$

Where C is independent of ε.

Proof. For t=1/2, $u(1/2) - w(1/2) = u(1/2) - u_0(1/2)$

$$= v_0(\tau) + w_0(\eta) + O(\sqrt{\varepsilon}) = O(\sqrt{\varepsilon})$$

For $1/2 < t < 1$, $L[u(t) - w(t)] = Lu(t) - Lw(t)$

$$= f(t) - f(t) = 0.$$

For t=1, $u(1) - w(1) = \phi_2 - \phi_2 = 0$.

Using maximum principle,

$$|u(t) - w(t)| \le C(||u(1/2) - w(1/2) + |u(1) - w(1)| + \sup|L u(s) - w(s)|) \text{ for s in } [1/2,1]$$

$$\le C\sqrt{\varepsilon}.$$

Hence the desired result.

IV Fitted Operator Method

On using central difference scheme and Bernoulli's generating function for the numerical solution of the problem (3.1)-(3.2) in the interval [0, ½], we have

$$L^h v_i \equiv -\sigma_1(\rho)\, \delta^2 v_i + b(t_i) v_i = f(t_i),\ i=1(1)\, N1-1, \quad (4.1)$$

$$v_0 = \phi_1, \quad v_{N1} = u_0(1/2) \quad (4.2)$$

Where $\sigma_1(\rho) = \sigma(-\rho\sqrt{b(0)})\, \sigma(\rho\sqrt{b(0)})$, $\rho = h/\sqrt{\varepsilon}$ and for (3.3)-(3.4) in the interval [1/2,1]

$$L^h w_i \equiv -\varepsilon\, \sigma_2(\rho)\, \delta^2 w_i + b(t_i) w_i = f(t_i),\ i=1(1)\, N2-1, \quad (4.3)$$

$$w_0 = u_0(1/2), \quad w_{N2} = \phi_2 \quad (4.4)$$

where $\sigma_2(\rho) = \sigma(-\rho\sqrt{b(1)}).\ \sigma(\rho\sqrt{b(1)})$, $\rho = h/\sqrt{\varepsilon}$, $t_{N1} = 1/2$, $t_{N2} = 1$ and $N1 + N2 = N$ intervals in [0,1].The schemes (4.1)-(4.2) and (4.3)-(4.4) are consistent with (3.1)-(3.2) and (3.3)-(3.4) as the step size h approaches zero. Bernoulli's function is defined as $\sigma(-x) = x/[1-\exp(-x)]$ and $\sigma(x) = x\exp(-x)/[1-\exp(-x)]$ for all x in [0,1].

The numerical solution u_i^h of (2.1)-(2.2) is defined as

$$u_i^h = v_i, \quad 0 \le t \le 1/2,$$

$$\text{and} \quad u_i^h = w_i, \quad 1/2 \le t \le 1.$$

The solutions v_i and w_i satisfy the stability result which is stated in the following theorem.

Theorem.4.1.. Let L^h be the operator in (4.1) and v_i be any smooth function then for all t in [0,1/2],

$$|v_i| \le C(|v_0| + |v_{N1}| + \sup|L^h v_i|),\ i=1(1)\, N1-1,$$

where C is independent of ε.

Proof. Refer[3].

Theorem.4.2.. Let L^h be the operator in (4.3) and w_i be any smooth function then for all t in [1/2,1],

$$|w_i| \le C(|w_0| + |w_{N2}| + \sup|L^h w_i|),\ i=1(1)\, N2-1,$$

where C is independent of ε.

Proof. Refer[3].

We have the error estimate $|u(t_i) - u_i^h|$ |n [0,1] as follows:

$$|u(t_i) - u_i^h| \le |u(t_i) - v(t_i)| + |v(t_i) - v_i| \quad \text{in } [0,1/2]$$

$$|u(t_i) - u_i^h| \le |u(t_i) - w(t_i)| + |wt_i) - w_i| \quad \text{in } [1/2,1]$$

Theorem.4.3.. Let v and v_i be the solutions of (3.1)-(3.2) and (4.1)-(4.2) in [0,1/2] then $|v(t_i) - v_i| \le C \min(\sqrt{\varepsilon}, h^2)$ in [0,1/2] where C is independent of i, h and ε.

Proof. Refer[3].

Theorem.4.4.. Let w and w_i be the solutions of (3.3)-(3.4) and (4.4)-(4.5) in [1 /2,1] then

$$|w(t_i) - w_i| \le C \min(\sqrt{\varepsilon}, h^2) \quad \text{in } [1/2,1]$$

where C is independent of i, h and ε.

Proof: Refer[3].

Following theorm gives the uniform and optimal convergence result which is the main result of this section.

Theorem.4.5.. Let u(t) and u_i^h be the solutions of (2.1)-(2.2) and (4.1)-(4.4) in [0,1] then

$$\| u(t_i) - u_i^h \| \le C \min(\sqrt{\varepsilon}, h^2)$$

where C is independent of i, h and ε.

Proof.Refer[3].

We have from the above Theorems 3.1, 3.2, 4.3 and 4.4 in [0,1],

$$|u(t_i) - u_i^h| \le |u(t_i) - v(t_i)| + |v(t_i) - v_i|$$
$$\le C[\sqrt{\varepsilon} + \min(\sqrt{\varepsilon}, h^2)] \quad \text{in } [0,1/2],$$
$$|u(t_i) - u_i^h| \le |u(t_i) - w(t_i)| + |wt_i) - w_i|$$
$$\le C[\sqrt{\varepsilon} + \min(\sqrt{\varepsilon}, h^2)] \quad \text{in } [1/2,1].$$

To find the error estimation, define in [0, 1]

$$\| u(t_i) - u_i^h \| = \max(\max\{|u(t_i) - u_i^h| \text{ in } [0,1/2]\}, \max\{|u(t_i) - u_i^h| \text{ in } [1/2,1]\}).$$

Using all the results derived above, we have

$$\| u(t_i) - u_i^h \| \le C[\sqrt{\varepsilon} + \min(\sqrt{\varepsilon}, h^2)]$$
$$\le C \min(\sqrt{\varepsilon}, h^2)$$

This is the desired result.

V FITTED MESH METHOD

In this section, the meshes are no longer uniform it is necessary to extend the fitted operator method from the uniform meshes in section 4 to non-uniform meshes [1]. To introduce the method of fitted mesh the problems discussed in the previous section is considered again here. In all cases a piecewise uniform mesh turns out to be sufficient for the construction of ε- uniform method. Of course more complicated meshes may also be used but the simplicity of the piecewise uniform meshes is considered. Furthermore piecewise uniform meshes turns out to be adequate for handling a surprisingly a wide variety of singularly perturbed problems,.

For linear reaction diffusion problem the following piecewise uniform mesh is constructed on the interval Ω= (0,1).Because there are boundary layers at two boundary points t=0 and t=1, the mesh should be condensing in a neighbourhood of each of these points. Two transition points are therefore required and mesh comprises three uniform pieces.

Choose a point τ satisfying $0<\tau \le 1/2$and assume that $N=2^r$, for some $r \ge 8$. The points τ and 1-τ are called transition points and divides Ω into three intervals (0,τ), (τ,1-τ) and (1-τ,1). The corresponding piecewise uniform mesh is constructed by dividing both (0,τ) and (1-τ,1) into N/4 equal subintervals and dividing the interval (τ,1-τ) into N/2 equal subintervals. Piecewise uniform meshes with N subintervals and a single parameter τ are denoted by Ω^r_N .

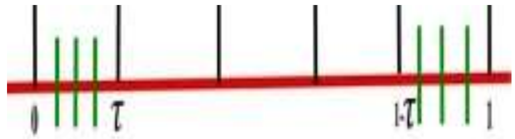

Fig 1. The piecewise uniform mesh Ω^r_N condensing at the point t=0 and t=1

The piecewise uniform mesh Ω^r_N is used with the following location of the transition point

$$\tau=\min\{1/4, 2\sqrt{\varepsilon} \qquad (5.1)$$

Depend on ε and N. This means location of the mesh points changes whenever ε or N changes. The transition point ï takes the value ¼ if N is exponentially large and so Ω^r_N will be a uniform mesh with N subintervals. This will happen rarely in practice. We are interested in real time situation in which for all other values of ï, 0<ï<1/4. the subintervals (0,τ) and (1-τ,1) are smaller than the subinterval (τ,1-τ).

To apply fitted scheme in the previous section, again subdivide the interval (τ,1-τ) into (τ,1/2) and (1/2,1-τ).

Therefore, in the interval (0,1/2) , take N/4 subintervals in (0,τ) and N/4 subintervals in (τ,1/2) and apply the scheme (4.1)-(4.2).

Find $\{v_\varepsilon\}_0^N \in R^{N+1}$, defined on Ω^r_N , such that $v_0 = \phi_1$, $v_N = u_0(1/2)$ and for all $1 \le i \le N$,

$$-\sigma_1(\rho)\ \delta^2 v_i + [1/12]\,(g_{i-1} + 10g_i + g_{i+1}) = 0\,.$$

Similarly, in the interval (1/2, 1), take N/4 subintervals in (1/2, 1-τ) and N/4 subintervals in (1-τ,1) and apply the scheme (4.3)-(4.4).

Find $\{w_\varepsilon\}_0^N \in R^{N+1}$, defined on Ω^r_N , such that $w_0 = u_0(1/2)$, $w_N = \phi_2$ and for all $1 \le i \le N$,

$$-\varepsilon\sigma_2(\rho\ \delta^2 w_i + [1/12]\,(g_{i-1} + 10g_i + g_{i+1}) = 0,\ i=1(1)\ N-1.$$

It should be noted that

$$\{u_\varepsilon\}_0^N = \{v_\varepsilon\}_0^N \text{ in } [0,1/2],$$
$$\{u_\varepsilon\}_0^N = \{w_\varepsilon\}_0^N \text{ in } [1/2,1].$$

The fitted mesh method discussed above is ε-uniform and the solution satisfies the ε-uniform error estimate, for all $N \ge N_0$, $0<\varepsilon\le 1$,

$$\sup \| u_{\varepsilon,N} - u_\varepsilon \|_\omega \le C\, N^{-1} \ln(N)\,,$$

Where N and C are independent of ε.

VI ALGORITHM

An algorithm is presented so that an user can perform experiment without any difficulty in steps.

Step 1: Subdivide the interval (0,1) into N intervals and generate a sequence $x_0, x_1, \ldots, x_N$.

Step 2: Subdivide the interval (0,1) into two subintervals (0,1/2) and (1/2,1)

Step 3: Subdivide the subintervals (0, 1/2) and (1/2, 1) into N/2 intervals of each.

Step 4: Rewrite the scheme (4.1)-(4.4) in tri-diagonal form

Step 5: Using sweep method rewrite the tri-diagonal form into a single step equation and solve for u_i.

Step 6: Apply the scheme (4.1)-(4.2) in the subintervals (0,1/2).

Step 7: Apply the scheme (4.3)-(4.4) in the subintervals (1/2,1).

Step 8: Subdivide the subinterval (0,1) into (0,τ), (τ,1-τ) and (1-τ, 1).

Step 9: Subdivide the interval (τ,1-τ) into (τ,1/2) and (1/2,1-τ).

Step 10: Subdivide all the four subintervals (0,τ), (τ,1/2), (1/2,1-τ) and (1-τ, 1) into N/4 intervals of each.

Step 11: Apply the scheme (4.1)-(4.2) in the subintervals (0,τ), (τ,1/2).

Step 12: Apply the scheme (4.3)-(4.4) in the subintervals (1/2,1-τ) and (1-τ, 1).

Using the steps 1-12, the numerical solution of u can be evaluated. If the user does not particular to view boundary layers then they can use the steps 1-7. And, If the user wish to view both the boundary layers then they can use the steps 1-12. Bernoulli's function with constant coefficient involved in the scheme (4.1)-(4.4) reduces both computation time and storage space in modern digital computers. The scheme is solved as a single step method. So the method is computationally faster.

VI I EXPERIMENTAL RESULT

To show the performance of the fitted operator and fitted mesh method and to view the uniform and optimal convergence experiments were performed using modern digital computers we consider a test problem frequently appear in aircraft optimal control guidance. The experimental result is presented in graphical form.

$$-\acute{\varepsilon}\, u'' + u = -[\cos^2(\pi t) + 2\pi^2 \cos(2\pi t)],\ t \in [0,1],\ u(0)=0,\ u(1)=0$$

Whose exact solution is given by

$$u(t) = [(e^{-(1-t)/\sqrt{\varepsilon}} + e^{-t/\sqrt{\varepsilon}})/(1 + e^{-1/\sqrt{\varepsilon}})] - \cos^2(\pi t).$$

Again on using fitted mesh method to the above problem we have the transition points τ and 1-τ as follows for N=16 and $\varepsilon = 10^{-4}$,

$$\tau = \min\{1/4, 2\sqrt{\varepsilon}\, n(N)\}$$
$$= \min\{1/4, 0.055451774\}$$
$$= \min\{0.25, 0.06)$$
$$= 0.06$$
$$\text{And } 1-\tau = 1-0.06 = 0.94.$$

The subintervals (0, 0.06), (0.06, 1/2), (1/2, 0.94) and (0.94, 1) are subdivided into N/4=4 subintervals each and so the interval (0, 0.06) with step 0.015, the interval (0.06, 1/2) with step size 0.11, the interval (1/2, 0.94) with step size 0.11 and the interval (0.94, 1) with step size 0.015. In the subintervals (0. 0.06) and (0.94, 1) more number of points can be achieved using fitted mesh method which cannot be done by fitted operator method directly.

From figure2 one can view the applicability of fitted mesh method via fitted operator method. At both ends of the boundary points one can view the layers so that a linear differential equation gives a nonlinear solution.

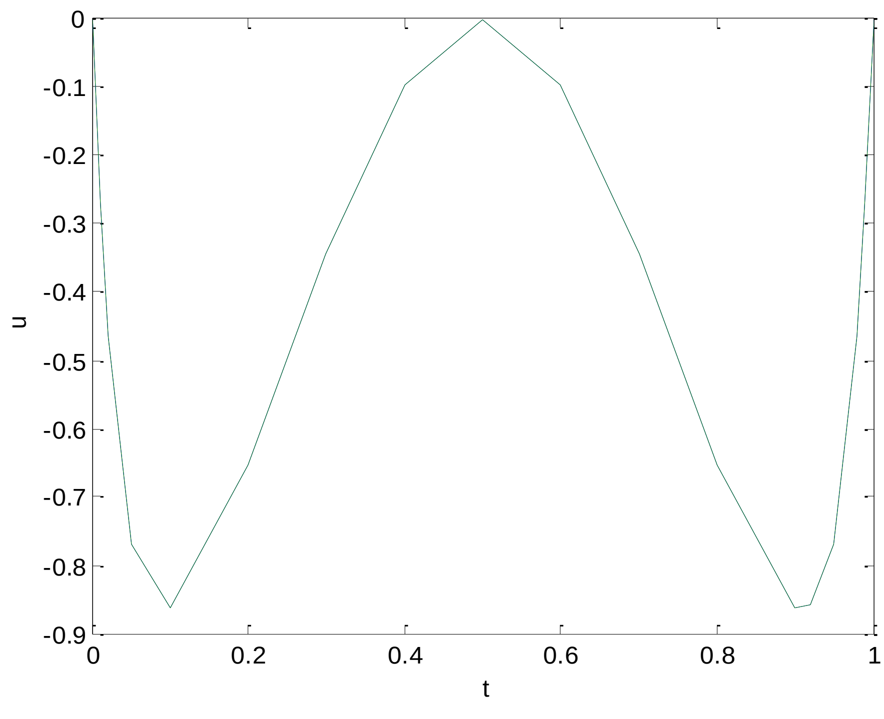

Figure 2

Using the sweep method, [21] the scheme is converted into a single step method. Because of it the computation time and storage space for the execution of the computer program get reduced considerably. No need for inversion of matrix for the evaluation of the numerical solution. From the above numerical experiment it is observed that the fitted operator and fitted mesh method proposed in this paper solves numerically the solution u both in the boundary layers and in smooth regions.

VIII CONCLUSION

In this paper a reaction diffusion problem with twin boundary layers is considered for the numerical solution. A fitted operator and a fitted mesh method are designed which take less time and storage space for the computation in modern digital computers. The method involved needs no iteration, no matrix inversion for the numerical convergence. It works as a single step method. The method is uniform and optimal and so the numerical solution reflects the properties of the exact solution of the problem to be solved for small values of the singular perturbation parameter.

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