APPROXIMATE BAYES ESTIMATE FOR THE PARAMETERS OF GENERALIZED COMPOUND RAYLEIGH DISTRIBUTION UNDER PRECAUTIONARY LOSS FUNCTION

[1] Parul Yadav and [2]Uma Srivastava
[1]Lecturer & [2]Professor
[1] GGIC Allahabad, Allahabad, UP 211001
[2] Department of Mathematics and Statistics
DDU Gorakhpur University, Gorakhpur, U.P.,India

Abstract: *The Generalized Compound Rayleigh Distribution is a special case of the three-parameter Burr type XII distribution. This unimodal hazard function is generalized and a flexible parametric model is thus constructed, which embeds the compound Rayleigh model, by adding shape parameter. The main objective of this paper is to determine the best estimator for the three parameter Generalized Compound Rayleigh Distribution assuming all the parameters unknown. The Methods under consideration are Maximum Likelihood Estimation and Bayesian Estimation under Precautionary Loss function. Lindley approximation is use to obtain the Approximate Bayes estimates of Generalized Compound Rayleigh Distribution assuming all the three parameters unknown under the PLF. We have studied the sensitivity of the Approximate Bayes Estimate of the model and presented a numerical study to illustrate the above technique on generated observations. The comparison is done by R-programming.*

Keywords: Maximum Likelihood Estimator, Bayesian Approximation, Precautionary Loss Function, Lindley Approximation.

I.Introduction

Mostert, Roux, and Bekker (1999) considered a gamma mixture of Rayleigh distribution and obtained the compound Rayleigh model with unimodal hazard function.
The Generalized Compound Rayleigh Distribution is a special case of the three-parameter Burr type XII distribution with probability density function (p.d.f.)

$$(x;\alpha,\beta,\gamma)=\alpha\gamma\beta^{\gamma}x^{\alpha-1}(\beta+x^{\alpha})^{-(\gamma+1)} \qquad x,\alpha,\beta,\gamma>0 \qquad (1.1)$$

With Probability Distribution Function

$$F(x)=1-(1-\beta x^{\alpha})^{-\gamma} \qquad x,\alpha,\beta,\gamma>0 \qquad (1.2)$$

Reliability function is

$$R(t)=\left(\frac{\beta}{\beta+t^{\alpha}}\right)^{\gamma}$$

Hazard rate function

$$H(t)=\alpha\gamma\frac{t^{\alpha-1}}{\beta+t^{\alpha}}$$

The Generalized compound Rayleigh model includes various well-known pdfs, namely

(i) Beta-Prime pdf (Patil, et al., 1984), if $\alpha=\beta=1$
(ii) $\alpha=1$
(iii) Burr XII pdf(Burr,1942), if $\beta=1$

Compound Rayleigh pdf (Siddiqui &Weiss,1963), if $\alpha=2$ Norstrom (1996) introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. These loss function approach infinitely near the origin to prevent underestimation and thus giving a conservative estimators , especially when , low failure rates are being estimated. These estimators are very useful and simple asymmetric precautionary loss function is

$$L(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^{2}}{\hat{\theta}} \qquad (1.3)$$

II. The Estimators

Let $x_1 \leq x_2 \leq \ldots\ldots\ldots \leq x_n$ be the n failures in complete sample case. The likelihood function is given by;

$$L(\underline{x}\,|\alpha,\beta,\gamma) = \prod_{j=1}^{n} f\,(x_j,\alpha,\beta,\gamma)$$

$$= \alpha^n \gamma^n \beta^{n\gamma} \prod_{j=1}^{n} x_j{}^{\alpha-1} \prod_{j=1}^{n} (\beta + x_j{}^{\alpha})^{-(\gamma+1)}$$

$$L(\underline{x}\,|\alpha,\beta,\gamma) = (\alpha\gamma)^n U e^{-\gamma T} \qquad (2.1)$$

Where

$$T = \sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{-\alpha}}{\beta}\right] \qquad \text{and} \qquad U = \prod_{j=1}^{n} \frac{x_j{}^{\alpha-1}}{(\beta + x_j{}^{\alpha})}$$

from equation(2.1) the log likelihood function is

$$Log\ L = n\ log\alpha + nlog\gamma + n\gamma\ log\beta + (\alpha-1)\sum_{j=1}^{n} logx_j$$

$$-(\gamma+1)\sum_{j=1}^{n}(\beta + x_j{}^{\alpha}) \qquad (2.2)$$

and differentiation of equation(2.2) with respect to α, β and γ yields respectively we get

$$\frac{\partial\ Log\ L}{d\alpha} = \frac{n}{\alpha} + \sum_{j=1}^{n} logx_j - \sum_{j=1}^{n} \frac{x_j{}^{\alpha} logx_j}{\beta + x_j{}^{\alpha}} - \gamma \sum_{j=1}^{n} \frac{x_j{}^{\alpha} logx_j}{\beta + x_j{}^{\alpha}} \qquad (2.3)$$

$$\frac{\partial\ Log\ L}{\partial\beta} = \frac{n\gamma}{\beta} - (\gamma+1)\sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}} \qquad (2.4)$$

can also be written as

$$\frac{\partial\ Log\ L}{\partial\beta} = -\sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}} + \gamma \sum_{j=1}^{n} \frac{x_j{}^{\alpha}}{\beta(\beta + x_j{}^{\alpha})} \qquad (2.5)$$

$$\frac{\partial\ Log\ L}{\partial\gamma} = \frac{n}{\gamma} + n\log\beta - \sum_{j=1}^{n} log\ \left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) - n\ log\ \beta$$

$$= \frac{n}{\gamma} - \sum_{j=1}^{n} log\ \left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) \qquad (2.6)$$

equating the expressions for the derivatives in (2.3)-(2.6) equal to zero and solving α, β and γ yield. The maximum likelihood estimators (MLE) of the parameters namely $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{\gamma}_{MLE}$.

However, no closed form solutions exist in this case the elimination of γ in $\frac{\partial\ Log\ L}{\partial\beta}$ and $\frac{\partial\ Log\ L}{\partial\alpha}$ and in $\frac{\partial\ Log\ L}{\partial\alpha}$ and $\frac{\partial\ Log\ L}{\partial\gamma}$ yield a set of equations in terms of β and α respectively.

$$\frac{\sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}}}{\sum_{j=1}^{n} \frac{x_j{}^{\alpha}}{\beta + x_j{}^{\alpha}}} - \frac{n}{\sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \qquad (2.7)$$

$$\frac{n}{\alpha} + \sum_{j=1}^{n} \log x_j - \Sigma \frac{x_j{}^{\alpha} logx_i}{(\beta + x_j{}^{\alpha})} - \frac{n \sum_{j=1}^{n} \frac{x_j{}^{\alpha} logx_i}{(\beta + x_j{}^{\alpha})}}{\sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \qquad (2.8)$$

Applying the Newton-Raphson method $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ can be derived and then from them $\hat{\gamma}_{MLE}$ can be obtained.

III. Bayes estimate for γ with known parameter α, β and γ

If $\hat{\alpha}$ and $\hat{\beta}$ is known we assume $\gamma(a,b)$ as conjugate prior for γ as

$$g(\gamma|\underline{x}) = \frac{b^a}{\Gamma a} \gamma^{a-1} e^{-\gamma b} \quad \text{(a,b)>0,}\gamma\text{>0} \qquad (3.1)$$

combining the likelihood function equation (2.1)and prior density equation (3.1) we obtain the posterior density of γ in the for

$$h(\gamma|\underline{x}) = \frac{\gamma^{n+a-1} e^{-\gamma(b+T)}}{\int_0^{\infty} \gamma^{n+a-1} e^{-\gamma(b+T)} d\gamma} \qquad (3.2)$$

Assuming;

$$\sum_{j=1}^{n} \log\left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) = T$$

$$h(\gamma|\underline{x}) = \frac{\gamma^{n+a-1} e^{-\gamma(b+T)} (b+T)^{n+a}}{\Gamma(n+a)} \qquad (3.3)$$

IV. Bayes Estimator under Precautionary Loss Function

The precautionary loss function is

$$\mathrm{L}(\Delta_1) = \mathrm{L}(\hat{u} - u) = \frac{(\hat{u}-u)^2}{\hat{u}} \tag{4.1}$$

$$\mathrm{E}_h\big(L(\Delta_1)\big) = \mathrm{E}_h\left(\frac{\hat{u}^2}{\hat{u}}\right) + \mathrm{E}_h(\hat{u}) - 2\mathrm{E}_h(u)$$

$$\frac{\partial \mathrm{E}_h(L(\Delta_1))}{\partial \hat{u}} = \mathrm{E}_h\left(-\frac{\mathrm{u}^2}{\hat{u}^2}\right) + 1 = 0$$

$$\Rightarrow \frac{-1}{\hat{u}^2}\mathrm{E}_\rho(\hat{u}^2) = -1$$

$$\Rightarrow \hat{u}_{BP} = \left[\mathrm{E}_\rho(u^2)\right]^{1/2} \tag{4.2}$$

Bayes estimate of γ under Precautionary loss function is

$$\hat{\gamma}_{BP} = [\mathrm{E}_h(\gamma^2)]^{1/2}$$

$$\mathrm{E}_h(\gamma^2) = \int_0^\infty \frac{\gamma^2 \gamma^{(n+a-1)} e^{-\gamma(b+T)} (b+T)^{(n+a)}}{\Gamma(n+a)} d\gamma \tag{4.3}$$

$$= \frac{(b+T)^{(n+a)}}{\Gamma(n+a)} \int_0^\infty \gamma^{(n+a+1)} e^{-\gamma(b+T)} dr$$

$$\hat{\gamma}_{BP} = \frac{[(n+a+1)(n+a)]^{1/2}}{(b+T)} \tag{4.4}$$

Approximate Bayes Estimators with unknown α, β and γ

Joint prior density α, β, γ is given by

$$\mathrm{G}\,(\alpha,\beta,\gamma) = g_1(\alpha) g_2(\beta) g_3(\gamma|\beta)$$

$$= \frac{c}{\delta\Gamma\xi} \beta^{-\xi} \gamma^{\xi-1} exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] \tag{4.5}$$

where

$$g_1(\alpha) = c \tag{4.6}$$

$$g_2(\beta) = \frac{1}{\delta} e^{-\frac{\beta}{\delta}} \tag{4.7}$$

$$g_3(\gamma) = \frac{1}{\Gamma\xi} \beta^{-\xi} \gamma^{\xi+1} e^{-\frac{1\gamma}{\beta}} \tag{4.8}$$

The Joint posterior with likelihood equation (2.2) and joint prior equation (4.5)

$$h^*(\alpha,\beta,\gamma) = \frac{\beta^{-\xi}\gamma^{\xi-1} exp\left[-\left(\frac{\gamma}{\beta}+\frac{\beta}{\delta}\right)\right] . L(\underline{x}|\alpha,\beta,\gamma)}{\int_\alpha \int_\beta \int_\gamma \beta^{-\xi}\gamma^{\xi-1} exp\left[-\left(\frac{\gamma}{\beta}+\frac{\beta}{\delta}\right)\right] . L(\underline{x}|\alpha,\beta,\gamma) d\alpha d\beta d\gamma} \tag{4.9}$$

The approximate Bayes estimators are evaluated as

$$\cup\,(\Theta) = \cup\,(\alpha,\beta,\gamma)$$

$$\hat{\cup}_{BS} = E\big(\cup\,|\underline{x}\big) = \frac{\int_\alpha \int_\beta \int_\gamma \cup(\alpha,\beta,\gamma) G^*(\alpha,\beta,\gamma) d\alpha d\beta d\gamma}{\int_\alpha \int_\beta \int_\gamma G^*(\alpha,\beta,\gamma) d\alpha d\beta d\gamma} \tag{4.10}$$

Lindley Approximation Procedure

The ratio of integrals in equation (4.9) does not seem to take a closed form so we must consider the Lindley approximation procedure as

$$E\big(\mu(\theta,p)\big|\underline{x}\big) = \frac{\int \mu(\theta).e^{(l(\theta)+\rho(\theta))} d\theta}{\int e^{(l(\theta)+\rho(\theta))}.d\theta} \tag{4.10a}$$

Lindley developed approximate procedure for evaluation of posterior expectation of $\mu(\theta)$. Several other authors have used this technique to obtain Bayes estimators (see Sinha(1986), Sinha and sloan(1988),Soliman(2001)).The posterior expectation of Lindley approximation procedure to evaluate of $\mu(\theta)$ in equation (4.9 and 4.10) under SELF, where where $\rho(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function (Lindley (1980)).

The modified form of equation (4.9) is given by

$$E\big(\cup\,(\alpha,\beta,\gamma|\underline{x}\big) = \cup\,(\Theta) + \frac{1}{2}(A \cup_1 \sigma_{11} + \cup_2 \sigma_{12} + \cup_3 \sigma_{13}) + B(\cup_1 \sigma_{21} + \cup_2 \sigma_{22} + \cup_3 \sigma_{23}) + P\,(\cup_1 \sigma_{31} + \cup_2 \sigma_{32} + \cup_3 \sigma_{33})] + (\cup_1 a_1 + \cup_2 a_2 + \cup_3 a_3 + a_4 + a_5) + 0\left(\frac{1}{n^2}\right) \tag{4.11}$$

Above equation is evaluated at MLE $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where

$$a_1 = \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13} \tag{4.12}$$

$$a_2 = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23} \tag{4.13}$$

$a_3 = \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33}$ (4.14)
$a_4 = \cup_{12}\,\sigma_{12} + \cup_{13}\,\sigma_{13} + \cup_{23}\,\sigma_{23}$ (4.15)
$a_5 = \frac{1}{2}(\cup_{11}\,\sigma_{11} + \cup_{22}\,\sigma_{22} + \cup_{33}\,\sigma_{33})$ (4.16)
and
$A = [\sigma_{11}l_{111} + \sigma_{12}l_{121} + 2\sigma_{13}l_{131} + 2\sigma_{23}l_{231} + \sigma_{22}l_{221} + \sigma_{33}l_{331}]$ (4.17)
$B = [\sigma_{11}l_{112} + 2\sigma_{12}l_{122} + 2\sigma_{13}l_{132} + 2\sigma_{23}l_{232} + \sigma_{22}l_{222} + \sigma_{33}l_{332}]$ (4.18)
$P = [\sigma_{11}l_{113} + 2\sigma_{13}l_{133} + 2\sigma_{12}l_{123} + 2\sigma_{23}l_{233} + \sigma_{22}l_{223} + \sigma_{33}l_{333}]$ (4.19)
To apply Lindley approximation on equation (4.9) we first obtain
$\sigma_{ij} = [-l_{ijk}]^{-1} i,j,k = 1,2,3$; where $l_{ijk}'s$ are respective partial derivatives of log likelihood
Likelihood function from Likelihood function(2.2) is

$$L = \alpha^n\gamma^n\beta^{n\gamma}\prod_{j=1}^{n} x_j^{\alpha-1}\prod_{j=1}^{n}(\beta + x_j^{\alpha})^{-(\gamma+1)}; (x,\beta,\gamma > 0)$$

$l_1 = \frac{n}{\alpha} + \sum_{j=1}^{n}\log x_j - (\gamma+1)\omega_{11}$ where $\omega_{11} = \sum_{j=1}^{n}\frac{x_j^{\alpha}\log x_j}{\beta + x_j^{\alpha}}$ (4.20)

$l_2 = \frac{n\gamma}{\beta} - (\gamma+1)\delta_{11}$ where $\delta_{11} = \sum_{j=1}^{n}\frac{1}{\beta + x_j^{\alpha}}$ (4.21)

$l_3 = \frac{n}{\gamma} + n\log\beta - q_{11}$ where $q_{11} = \sum_{j=1}^{n}\log(\beta + x_j^{\alpha})$ (4.22)

$l_{11} = \frac{-n}{\alpha^2} - \beta(\gamma+1)\omega_{122}$ where $\omega_{122} = \sum_{j=1}^{n}\frac{x_j^{\alpha}(\log x_j)^2}{(\beta + x_j^{\alpha})^2}$ (4.23)

$l_{22} = \frac{-n\gamma}{\beta^2} - (\gamma+1)\delta_{12}$ where $\delta_{12} = \sum_{j=1}^{n}\frac{1}{(\beta + x_j^{\alpha})^2}$ (4.24)

$l_{33} = \frac{\partial^2 L}{\partial\gamma^2} = \frac{-n}{\gamma^2}$ (4.25)

$l_{12} = (\gamma+1)\omega_{14} = l_{21}$ where $\omega_{14} = \Sigma\frac{x_j^{\alpha}\log x_j}{(\beta + x_j)^2}$ (4.26)

$l_{13} = -\sum_{j=1}^{n}\frac{x_j^{\alpha}\log x_j}{\beta + x_j^{\alpha}} = -\omega_{11} = l_{31}$ (4.27)

$l_{23} = \frac{n}{\beta} - \sum_{j=1}^{n}\frac{1}{\beta + x_j^{\alpha}} = \frac{n}{\beta} - \delta_{11} = l_{32}$ (4.28)

$l_{111} = \frac{2n}{\alpha^3} + (\gamma+1)\beta\ \omega_{133}$,where $\omega_{133} = \sum_{j=1}^{n}\frac{x_j^{\alpha}(\log x_j)^3(\beta - x_j^{\alpha})}{(\beta + x_j^{\alpha})^3}$ (4.29)

$l_{222} = \frac{2n\gamma}{\beta^3} - 2(\gamma+1)\ \delta_{13}$ where $\delta_{13} = \sum_{j=1}^{n}\frac{1}{(\beta + x_j^{\alpha})^3}$ (4.30)

$l_{333} = \frac{\partial^3 L}{\partial\alpha^3} = \frac{2n}{\gamma^3}$ (4.31)

$l_{112} = -(\gamma+1)\ \omega_{123}$ where $\omega_{123} = \sum_{j=1}^{n}\frac{x_j^{\alpha}(\log x_j)^2(\beta - x_j^{\alpha})}{(\beta + x_j^{\alpha})^3}$ (4.32)

$l_{112} = l_{121} (\because l_{12} = l_{21})$
$l_{113} = -\beta\sum_{j=1}^{n}\frac{(\log x_j)^2 x_j^{\alpha}}{(\beta + x_j^{\alpha})^2} = -\beta\ \omega_{122}$ (4.33)

$l_{221} = -2(\gamma+1)\omega_{113}$ where $\omega_{113} = \sum_{j=1}^{n}\frac{x_j^{\alpha}\log x_j}{(\beta + x_j^{\alpha})^3}$ (4.34)

$l_{223} = \frac{-n}{\beta^2} + \delta_{12}$ (4.35)
$l_{331} = 0 = l_{313}$ (4.36)
$l_{332} = 0 = l_{323}$ (4.37)
The matrix of derivatives is given as

$$[-l_{ijk}] = -\begin{bmatrix} l_{111} & l_{112} & l_{113} \\ l_{221} & l_{222} & l_{223} \\ l_{331} & l_{332} & l_{333} \end{bmatrix}$$

$$[-l_{ijk}] = \begin{bmatrix} \frac{2n}{\alpha^3} + (\gamma+1)\beta\omega_{133} & -(\gamma+1)\omega_{123} & -\beta\omega_{122} \\ -2(\gamma+1)\omega_{113} & \frac{2n\gamma}{\beta^3} - 2(\gamma+1)\delta_{13} & -\frac{n}{\beta^2} + \delta_{12} \\ 0 & 0 & \frac{2n}{\gamma^3} \end{bmatrix}$$

$$[-l_{ijk}] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Determinant of = $-[l_{ijk}]$

$$D= -M_{33}\{M_{11}M_{22} - M_{12}M_{21}\} \quad (4.38)$$

$$[-l_{ijk}]^{-1} = \frac{(Adjoint\ of[-l_{ijk}])'}{[-l_{ijk}]}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} -\frac{M_{22}M_{33}}{D} & \frac{M_{12}M_{33}}{D} & \frac{M_{22}M_{13} - M_{12}M_{23}}{D} \\ \frac{M_{21}M_{33}}{D} & -\frac{M_{11}M_{33}}{D} & \frac{M_{11}M_{23} - M_{21}M_{13}}{D} \\ 0 & 0 & \frac{M_{11}M_{22} - M_{12}M_{21}}{D} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} Y_{11/D} & Y_{12/D} & Y_{13/D} \\ Y_{21/D} & Y_{22/D} & Y_{23/D} \\ 0 & 0 & Y_{33/D} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad (4.39)$$

Approximate Bayes Estimator under Precautionary Loss function

$$\hat{\mu}_{ABP} = [E_h(U^2)]^{1/2} \quad (4.40)$$

where

$$\widehat{U}_{ABP} = \frac{\int_\alpha \int_\beta \int_\gamma U^2 G^*(\alpha,\beta,\gamma) d\alpha d\beta d\gamma}{\int_\alpha \int_\beta \int_\gamma G^*(\alpha,\beta,\gamma) d\alpha d\beta d\gamma}$$

The above equation (4.40) is evaluated by method of Lindley approximation whose simplified form is equation (4.10)

$\widehat{U}_{ABP} = E(U|\underline{x})$

$E(U|\underline{x}) = U^2 + \varphi_1 + \varphi_2$

$$E(U|\underline{x}) = U + (U_1a_1 + +U_2a_2 + U_3a_3 + a_4 + a_5) + \frac{1}{2}\Big[A(U_1\sigma_{11} + +U_2a_{12} + U_3a_{13} + B(U_1\sigma_{21} + U_2\sigma_{22} + U_3\sigma_{23}) + P(U_1\sigma_{31} + U_2\sigma_{32} + U_3\sigma_{33}) + 0.\left(\frac{1}{n^2}\right)\Big]$$

evaluated at the MLE $\widehat{U} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$

V. Special cases

1. Approximate Bayes estimate of α

$U = \alpha^2$

$\varphi_1 = U_1\, a_1 + U_2\, a_2 + U_3\, a_3 + a_4 + a_5$

$$\varphi_1 = 2\alpha\left(\left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{12}}{D} + \left(\frac{\xi-1}{\gamma} + \frac{-1}{\beta}\right)\frac{Y_{13}}{D}\right) + \sigma_{11},$$

$$= -\frac{1}{2}[(A\sigma_{11} + B\sigma_{21})2\alpha]$$

$\varphi_2 = \alpha[(A\sigma_{11} + B\sigma_{21})]$

$$E(\alpha^2|\underline{x}) = \alpha^2 + \alpha\left\{2\left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{12}}{D} + \left(\frac{\xi-1}{\gamma} + \frac{1}{\beta}\right)\frac{Y_{13}}{D} + (A\sigma_{11} + B\sigma_{21})\right\} + \sigma_{11}$$

$E(U|\underline{x}) = \alpha^2 + \alpha\varphi_3 + \sigma_{11}$

$$\hat{\alpha}_{ABP} = [\alpha^2 + \alpha\varphi_3 + \sigma_{11}]^{\frac{1}{2}}\ ; \quad \text{at } (\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\gamma}_{ML}) \quad (5.1)$$

Where

$$\varphi_3 = 2\left[\left\{\left(\left(\frac{-\xi}{\beta}+\frac{\gamma}{\beta^2}-\frac{1}{\delta}\right)\frac{Y_{12}}{D}-\left(\frac{\xi-1}{\gamma}+\frac{1}{\beta}\right)\frac{Y_{13}}{D}\right)\right.\right.$$
$$+\sigma_{11}\left(\frac{Y_{11}}{D}\left(\frac{2n}{\alpha^3}-(\gamma+1)\omega_{133}+2\frac{Y_{12}}{D}(\gamma+1)\omega_{123}-2\frac{Y_{13}}{D}\beta\omega_{122}-2Y_{22}(\gamma+1)\omega_{113}\right)\right.$$
$$\left.\left.\left.+\sigma_{21}\left(-\frac{Y_{11}}{D}(\gamma+1)\omega_{123}-4\frac{Y_{12}}{D}(\gamma+1)\omega_{113}+2\frac{Y_{13}}{D}\omega_{112}+\frac{Y_{22}}{D}\left(\frac{2n\gamma}{\beta^3}-2(\gamma+1)\delta_{13}\right)\frac{2Y_{23}}{D}\left(\frac{-n}{\beta^2}+\delta_{12}\right)\right)\right)\right\}\right]$$

2. Approximate Bayes Estimate of $\boldsymbol{\beta}$

$\cup\,(\alpha,\beta,\gamma) = \cup$
$\cup = \beta^2$
$E(\cup\,|\underline{x}) = \beta^2 + \varphi_1 + \varphi_2$
$\varphi_1 = \cup_1\, a_1 + \cup_2\, a_2 + \cup_3\, a_3 + a_4 + a_5$
$\varphi_1 = \beta\left(2\left(\frac{-\xi}{\beta}+\frac{\gamma}{\beta^2}-\frac{1}{\delta}\right)\frac{Y_{22}}{D}+2\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)\frac{Y_{23}}{D}+\sigma_{22}\right)$

$$\varphi_2 = \frac{1}{2}[\cup_1\,(A\sigma_{11}+B\sigma_{21}+P\sigma_{31})+U_2(A\sigma_{12}+B\sigma_{22}+P\sigma_{32})+U_3(A\sigma_{13}+B\sigma_{23}+P\sigma_{33})]$$

$\varphi_2 = \beta(A\sigma_{12}+P\sigma_{22}).$
$E(\cup\,|\underline{x}) = \beta^2 + \beta\left(2-\left(\frac{-\xi}{\beta}+\frac{\gamma}{\beta^2}-\frac{1}{\delta}\right)\frac{Y_{22}}{D}+2\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)\frac{Y_{23}}{D}+A\sigma_{12}+B\sigma_{22}\right)+\sigma_{22}$
$E(\cup\,|\underline{x}) = (\beta^2+\beta\varphi_4+\sigma_{22}).$
$\hat{\beta}_{ABP} = [\beta^2+\beta\varphi_4+\sigma_{22}]^{\frac{1}{2}}$; at $(\hat{\alpha}_{ML,}\hat{\beta}_{ML,}\hat{\gamma}_{ML})$ (5.2)

$$\varphi_4 = 2\left[\left(\frac{-\xi}{\beta}+\frac{\gamma}{\beta^2}-\frac{1}{\delta}\right)\frac{Y_{22}}{D}-\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)\frac{Y_{23}}{D}\right.$$
$$+\sigma_{12}\left(\frac{Y_{11}}{D}\left(\frac{2n}{\alpha^3}-(\gamma+1)\omega_{133}\right)+2\frac{Y_{12}}{D}(\gamma+1)\omega_{123}-\frac{2Y_{13}}{D}\beta\omega_{122}-\frac{2Y_{22}}{D}(\gamma+1)\omega_{113}\right)$$
$$\left.+\sigma_{22}\left(-\frac{Y_{11}}{D}(\gamma+1)\omega_{123}-4\frac{Y_{12}}{D}(\gamma+1)\omega_{113}+2Y_{13}\omega_{112}+2Y_{23}\left(\frac{-n}{\beta^2}+\delta_{12}\right)+Y_{22}\left(\frac{2n\gamma}{\beta^3}-2(\gamma+1)\delta_{13}\right)\right)\right]$$

3. Approximate Bayes estimate of $\boldsymbol{\gamma}$

$\cup\,(\alpha,\beta,\gamma) = \cup$
$\cup = \gamma^2$
$E(u|\underline{x}) = \gamma^2 + \varphi_1 + \varphi_2$
$\varphi_1 = \cup_1\, a_1 + \cup_2\, a_2 + \cup_3\, a_3 + a_4 + a_5$
$\varphi_1 = 2\gamma\frac{Y_{33}}{D}\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)+\sigma_{33}$
$\varphi_2 = \frac{1}{2}[\cup_1\,(A\sigma_{11}+B\sigma_{21}+P\sigma_{31})+\cup_2\,(A\sigma_{12}+B\sigma_{22}+P\sigma_{32})+\cup_3\,(A\sigma_{13}+B\sigma_{23}+P\sigma_{32})]$
$\varphi_2 = \frac{1}{2}.2\gamma(A\sigma_{13}+B\sigma_{23}+B\sigma_{33})$
$E(\cup\,|\underline{x}) = \gamma^2+\gamma\left[2\frac{Y_{33}}{D}\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)+A\sigma_{13}+B\sigma_{23}+P\sigma_{33}+\sigma_{33}\right]$
$= \gamma^2+\gamma\varphi_5+\sigma_{33}$
$\hat{\gamma}_{ABP} = [E(\cup\,|\underline{x})]^{\frac{1}{2}}$
$\hat{\gamma}_{ABP} = [\gamma^2+\gamma\varphi_5+\sigma_{33}]^{\frac{1}{2}}$; at $(\hat{\alpha}_{ML,}\hat{\beta}_{ML,}\hat{\gamma}_{ML})$ (5.3)
Where

$$\varphi_5=\left[2\frac{Y_{23}}{D}\left(\frac{\xi-1}{\gamma}-\frac{1}{\beta}\right)+\sigma_{13}\left(\frac{Y_{11}}{D}\left(\frac{2n}{\alpha^3}-(\gamma+1)\omega_{133}\right)+2\frac{Y_{12}}{D}(\gamma+1)\omega_{123}-\frac{2Y_{13}}{D}B\omega_{122}-\frac{2Y_{22}}{D}(\gamma+1)\omega_{113}\right)+\sigma_{23}\left(-\frac{Y_{11}}{D}(\gamma+1)\omega_{123}-4\frac{Y_{12}}{D}(\gamma+1)\omega_{113}+2\frac{Y_{13}}{D}\omega_{112}+2Y_{23}\left(\frac{-n}{\beta^2}+\delta_{12}\right)+Y_{22}\left(\frac{2n\gamma}{\beta^3}-2(\gamma+1)\delta_{13}\right)\right)+\sigma_{33}\left(2\frac{Y_{12}}{D}\omega_{112}-\frac{Y_{11}}{D}\beta\omega_{122}+\frac{Y_{22}}{D}\left(-\frac{n}{\beta^2}+\delta_{12}\right)+\frac{2Y_{33}}{D}\frac{n}{\gamma^3}\right)\right]$$

Simulation and Numerical Comparison

The simulations and numerical calculations are done by using R Language programming and results are presented in form of tables in table (1).

1. The Random variable of Generalized Compound Rayleigh Distribution is generated by R-Language programming by taking the values of the parameters α,β,γ , taken as $\alpha=1$, $\beta=0.5$ and $\gamma=0.8$ in the equations[(4.41)-(4.43)] and equation(2.2).
2. Taking the different sizes of samples n=10(10)80 with complete sample, MLE's, the Approximate Bayes estimators ,and their respective MSE's (in parenthesis) are obtained by repeating the steps 500 times, are presented in the table (1), for t=0.5, k= -10, R(t)=0.42, H(t)=0.625 and parameters of prior distribution a =2 and b =3.
3. Table (1) presents the MSE of α and β and Approximate Bayes estimators of α and β (for α,β and γ unknown) under LLF and PLF. The MSE's in all above cases are presented in parenthesis. Here $\hat{\alpha}_{ABL}$ *and* $\hat{\beta}_{ABL}$ under LLF has the lowest MSE's which shows its domination amongst other estimators.
4. Table (1) presents the MLE of parameter γ of (for known α,β) and Approximate Bayes estimator of γ under Precautionary Loss function(PLF) (for α,β and γ unknown). The MSE's in all above cases are presented in parenthesis. The estimators have minimum MSE's for small sample sizes, as the sample sizes increase, the MSE's increased. Among all the four estimators $\hat{\beta}_{ABP}$ under PLF has the lowest MSE.

Table (1)
Mean and MSE'S of α,β,γ
($\alpha=1$, $\beta=0.5$ and $\gamma=0.8$)

n	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{ABP}$	$\hat{\beta}_{ML}$	$\hat{\beta}_{ABP}$	$\hat{\gamma}_{ML}$	$\hat{\gamma}_{BP}$	$\hat{\gamma}_{ABP}$
10	0.6845795	0.7125461	0.6584213	0.6125632	0.5900124	0. 74576451	0. 843621
	[0.886547]	**[0.800450]**	**[0.052146]**	**[0.004478]**	**[0.0256142]**	**[4.054x10^{-2}]**	**[2.5522x10^{-2}]**
20	0.6998754	0.8007956	0.6231444	0.5845165	0.6922601	0. 750967	0. 852534
	[0.895647]	**[0.825804]**	**[0.852365]**	**[0.003208]**	**[0.026537]**	**[3.6788x10^{-2}]**	**[4.2797 x10^{-2}]**
30	0.7451688	0.8123467	0.5825888	0.5787745	0.6865478	0. 755524	0. 875134
	[0.057125]	**[0.845796]**	**[0.658458]**	**[0.082335]**	**[0.096548]**	**[0.4007528]**	**[0.6458796]**
40	0.79865421	0.8897454	0.5482658	0.5874515	0.86975682	0. 8482655	0. 923851
	[0.457812]	**[0.001415]**	**[0.002354]**	**[0.001257]**	**[0.003265]**	**[0.8956472]**	**[0.8529874]**
50	0.8956874	0.9710110	0.5996584	0.588925	0.8490011	0.8710110	0.8770789
	[0.004578]	**[0.001725]**	**[0.004577]**	**[0.025874]**	**[0.004265]**	**[0.001725]**	**[0.001522]**

60	0.91478523	0.9978542	0.6990011	0.6700415	0.9490011	0.9710110	0.9254879
	[0.004325]	**[0.001124]**	**[0.054663]**	**[0.055412]**	**[0.004226]**	**[0.001725]**	**[0.001522]**
70	1.000001	0.9888757	0.6988845	0.6859574	0.9454543	0.9814406	0.9956428
	[0.000125]	**[0.004221]**	**[0.001125]**	**[0.032147]**	**[0.001367]**	**[0.004221]**	**[0.038249]**
80	1.2354782	1.1478253	0.7235814	0.70225413	0.9657432	1.0147406	1.036426
	[0.325874]	**[0.045821]**	**[0.025258]**	**[0.3258145]**	**[0.001245]**	**[0.004231]**	**[0.034001]**

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