A STUDY ON ISOPERIMETRIC INEQUALITY

[1]ALKA BENNY, [2]REMONA BASTIN, [3]RHEA T.C, [4]RITA RIYA D.J, [5]SAYEEDHA P.S, [6]SNEHA JAYAKUMAR
[1]ASSISTANT PROFESSOR, [2-6]STUDENT
DEPARTMENT OF MATHEMATICS
ST TERESA'S COLLEGE (AUTONOMOUS), ERNAKULAM, KOCHI-682 001, KERALA, S.INDIA.

Abstract: Isoperimetric literally means having equal perimeters, used especially of geometrical figures having a constant scale, used of a line on a map. In Mathematics the Isoperimetric Inequality is a geometric inequality involving the surface area of a set and its volume. In this paper we have made a brief study on Isoperimetric Inequality in which we have included history, proofs and application of Isoperimetric Inequality. In the study we have also made a specification on "Why we consider a circle with a given perimeter encloses the greatest area among all planar region.

Index Terms **– Geometric Figures, Geometrical Inequality, Isoperimetric Inequality, Planar Region.**

I. INTRODUCTION

The Isoperimetric inequality theorem states that among all planar regions with a given perimeter P, the circle encloses the greatest area. This result which is also known as the isoperimetric inequality, dates back to antiquity .The theorem has generalization to higher dimensions and even has many variants in two dimensions , for example one version states that among all polygons with K sides and a fixed perimeter ,those are perfectly symmetric (i.e. Regular) have the greatest area.

This area and volume optimization theorem is especially appealing because they offer physical insights into nature. They tell us why a cat curls upon a cold winter night to minimize its exposed surface area. They help us understand why honey bees build hives with cells that are perfectly hexagonal in shape. The Isoperimetric inequality also helps us explain why water pipes should have round cross section. Of course, nature is complicated and the underlying mathematics can be difficult.

In three dimension the sphere has the greatest volume for a given surface area.

II. HISTORY

The remarkable Isoperimetric inequality even has a literary history dating back some twenty-one century to Virgil's Aeneid and the saga of Queen Dido .Apparently, the good Queen had more than her fair share of entrepreneurial skill and mathematical ability as well as misfortune of epic proportion. Her legend recounts, among other tragedies, the murder of her father by her brother, who then directed his intentions towards her. She was obliged to assemble her valuables and flee her native city of Tyria in ancient Phoenicia. In due course, her ship landed in North Africa, where she made the following offers to a local chieftain .In return for her fortune, she would be ceded much land as she could isolate with the skin of an ox. The proportion must have seemed too good to refuse. It was agreed to ,a large ox was sacrificed for its hide .Queen Dido broke it down into extremely thin strips of leather, which she tied together to construct a giant semicircle that, when combined with the natural boundary imposed by the sea, turned out to encompass far more area than anyone might have imagined. And upon this land, the city of carthage was born. Evidently, she knew the Isoperimetric inequality and understood how to use this fact to find the best solution to her problem, which uses a semicircle rather than a circle.

III. DEFINITION

In mathematics, the Isoperimetric Inequality is a geometrical inequality involving the surface of a set and its volume .It literally means "having the same perimeter".

IV. FORMULATION

Let $c(t) = (x(t),y(t))$ be a simple ,closed ,positively oriented and regular parameterized curve with t belongs (a,b).Denote the area enclosed in the above defined curve c(t) with A .For a given length of $c(t) = (x(t),y(t))$ we have

$$l^2 - 4\pi A \geq 0$$

or equivalently $A \leq l^2/4\pi$ with equality if and only if c(t) is a circle.

V. TWO EQUIVALENT STATEMENTS

For closed curves in a plane,

A) Of all such curves with a fixed perimeter the curve that forms a circle encloses the greatest area.
B) Of all such curves enclosing a fixed area , the curve that forms a circle has the shortest perimeter.

It can be shown that if we assume A , then we can show B and vice versa.

VI. ZENODORUS

Zenodorus was an ancient Greek mathematician from around 200 B.C .His most important work was on isometric figures ,which has sadly been lost ,in which Zenodorus examined figures with equal perimeters but different shapes. Parts of his work survived through references by other mathematicians such as Pappus and Theon of Alexandria.

VII. PROOFS AND THEOREMS

Zenodorus managed to prove many important statements which suggested the isoperimetric problem, but the mathematics of that time was not advanced enough to prove the problem itself. Despite these limitations Zenodorus still proved:

A) The regular polygon with most angles had the greatest area.
B) The circle has greatest area than any regular polygon of equal perimeter.
C) The equilateral and equiangular (in other words ,regular) polygon has the greatest area of any polygon with same perimeter and number of sides.

He also theorized that sphere has the greatest volume of any solid surface with the same surface area which is the answer to the 3D isoperimetric problem.

VIII. DIDO'S PROBLEM

Dido's problem is one of the most famous math problems of antiquity. In literature it was first noted in Virgil's Aeneid. The story goes that queen Dido was chased away from her home in Pheonicia by her brother .She went to Africa and made an agreement with the natives to purchase a piece of land which she could enclose with a bull's hide.

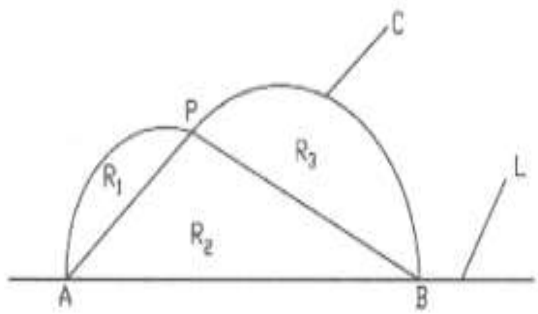
IX. STEINER'S PROOF

It was not until the 19th century that a proof was found for classical isoperimetric problem. Jakob Steiner, a swiss mathematician led the way with his four-hinge method in 1838. As noted above, Steiner did not succeed in a following proof as he was not above to prove the existence of a solution. Carathedory completed Steiner's proof.

Steiner's proof relies on two facts that are readily accessible at high school level.

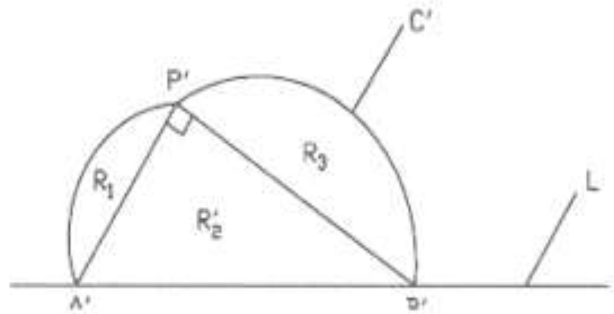
A) Any inscribed triangle in a circle with a diameter as its hypotenuse has a right angle between its leg.
B) Right triangles have the largest area of any triangle with the same legs. This last point is obvious, given legs X and Y with angle θ between them**: A= ½ XY sin(θ)** which is maximized for **sin (θ) = 1**

   Suppose we have the following **Initial curve.**



Steiner symmetrization works by choosing a point P and forming a triangle between that point and the endpoints on the line of the semicircle. As can be seen above, AP, PB, AB form a triangle. Let us then move the points A and B on the line such that angle APB is a right angle. By doing we so we keep the lengths AP and BP the same. Thus from above, out new triangle is bigger than our old one.

If we divided up the initial curve into the area inside the triangle, R_2 and then the area in other parts of the curve, R_1 and R_3 the area bounded by the curve after this symmetrization is larger. R_1 and R_3 stay the same but $R_2' > R_2$. So the sum of the areas is greater.



Result of first iteration

As the number of iterations increases the curve tends towards a circle which would be the result of an infinite number of iterations. In each iteration the area bounded by the curve can never decrease. Thus we conclude that a circle has the greatest internal area for its measured perimeter.

As noted above, this proof only shows the uniqueness of the circle as the solution. It does not show existence, as noted by the German mathematician Peter Dirichlet. That is it remains to be shown that there is a curve that maximizes area with respect to the perimeter

X. THE FIRST RIGOROUS PROOF

While Steiner had made a significant contribution to the understanding of the isoperimetric problem his proof was not complete. The first complete proof was developed by Weierstrass using variational calculus. This method was also used to prove higher dimensional versions of the isoperimetric problem. Schwartz used Steiner symmetrization to show the 3D case in 1884.

For higher dimensions the problem can be stated: Maximize the hyper volume with respect to the hyper surface area. Below is the outline of the 2D proof using variational calculus.

Proof:

Define the plane curve C as :

$$r(t)=(x(t),y(t))$$

We can then define the area and length

$$A=\frac{1}{2}\int_0^T\left(y(t)\frac{dx}{dt}-x(t)\frac{dy}{dt}\right)dt=\int_0^T F(t)dt$$

$$L=\int ds=\int_0^T\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^T G(t)dt$$

We can then define

$$H(t) = F(t) + \lambda\, G(t)$$

Where λ is the Lagrange multiplier.

We get two Euler – Lagrange equation:

$$\frac{\partial H}{\partial x}-\frac{d}{dt}\left(\frac{\partial H}{\partial x_t}\right)=0$$

$$\frac{\partial H}{\partial y}-\frac{d}{dt}\left(\frac{\partial H}{\partial y_t}\right)=0$$

When x_t and y_t is the time derivative of x and y respectively. Solving this system of equations gives the solution:

$$(y(t)-C_1)^2 + (x(t-)C_2)^2 =\lambda^2$$

For which C1 and C2 are constants of integration. The form of this equation is that of a circle with radius λ

XI . PAPPUS' BEES

Pappus' was one of the latest classical mathematicians following Zenodorus by 300 years putting Pappus' somewhere between 100 A.D. AND 200 A.D. It is through Pappus' work, as well as other mathematical works , which was unfortunately lost to history.

Pappus' is most famous for his analysis of the hexagonal structure of the honeycomb in a bee hive and its relationship to the isoperimetric problem titled On the Sagacity of Bees .

The surface of a plane can only be tiled by three regular polygon shapes. The equilateral triangle, square and hexagon . This fact has to do with the fact that the external angle is a whole number multiple of the internal angle for only these three shape.

The total internal angle for a polygon with n sides is 180(n-2) . We consider only regular polygons using the result of Zenodorus proof of (3) above.

n	Internal angle sum	External angle	Internal angle	$\frac{\text{External angle}}{\text{Internal angle}}$
3*	**180**	**300**	**60**	**5***
4*	**360**	**270**	**90**	**3***
5	**540**	**252**	**108**	**2.333**
6*	**720**	**240**	**120**	**2***
7	**900**	**231.429**	**128.571**	**1.8**
8	**1080**	**225**	**135**	**1.667**

Table 1: Table of internal and corresponding External angles for regular polygons of n side. The starred entries are those which have an integer multiple of the internal angle for the external angle. This means that these polygons can be tiled on a plane.

Thus we get the following possible tilings:

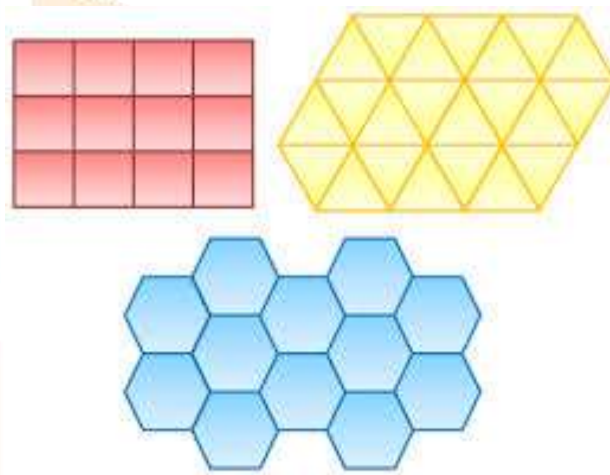


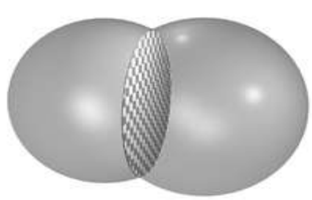
Figure 4: The three only possible tilings on the surface of a plane using regular polygons.

The regular triangle, square and hexagon all tile the plane without gaps . However Zenodorus showed in (1) that the regular polygon with the most angles has the most area. From these three regular polygons, the hexagon has the most angles , and thus it has the most efficient shape for tiling the surface of a plane .

What is interesting, both in modern times and what Pappus made note of , is that bees figured this optimization out. In this case nature has an example of this the most efficient tiling which takes into account the fact that there are only three regular polygons which can tile a plane as well as Zenodorus observation that the regular polygon with the most angles bounds the greatest area.

XII. DOUBLE BUBBLE CONJUCTURE

The double bubble is the surface in R_3 obtained by taking two pieces of round spheres separated by a flat disk meeting along a single circle at an angle of $2\pi/3$.



It has long been thought that the double bubble minimizes area among all piece wises smooth surfaces enclosing two equal volumes. Experimental evidence towards this conjecture can be obtained by blowing soap bubbles of equal size and pushes them together until they conglomerate to form a compound bubble , one obtains a double bubble . Such experiments we carried out by the Belgian physicist J. Plateau in the middle of the 19^{th} century. Plateau established experimentally that soap bubble cluster is a piecewise smooth surface having only two types of singularities. The first singularity occurs when 3 smooth surfaces come together along a smooth triple curve at angle of 120°. The second type of singularity occurs when 6 smooth surfaces and 4 triple curves converge at a point, with all angles equal. The angles are equal to those of the cone over the 1-skelton of a a regular tetrahedron. C.V Boys , discussing the work of plateau in his famous book on a soap bubbles [5] writes ,"when however the bubble is not single , say two have been blown in real contact with one another ,again the bubbles must together take such a form that the total surface of the two spherical segments and of the part common to both , which shall call the interface, is the smallest possible surface which will contain the two volumes of air and keep them, separate ". We have obtained a proof of this conjecture for the case of two equal volumes.

References

[1] [Blaschke] Blaschke, Wilhelm. Die Grundlehren der Mathematischen Wissenschaften. Berlin: Springer Verlag, 1950. Print.

[2] [Carmo] Carmo, do Manfredo P. Dierentialgeometrie von Kurven und Flächen. Trans. Michael Grüter. Wiesbaden: Vieweg Verlag, 1993. Print.

[3] [Froehlich] Froehlich, Steen. Konvexe Kurven und das Isoperimetrische Problem. FU Berlin, 2009. Web. 7 Feb 2013.