SOME SUBSPACE OF ENTIRE RATE SEQUENCE OF INTERVAL NUMBERS

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Abstract : In this paper we introduced the new concept of interval valued sequence space $\Gamma_{\lambda,\pi}(IR)$ where (π_k) is a sequence of positive numbers. We present the different properties like completeness, solidness, AB property etc.

IndexTerms - **Banach space, AB space, AK property, interval numbers**

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I. Introduction

Mathematical structures in general, have been constructed with real or complex numbers. In recent years, these mathematical structures are replaced by fuzzy numbers or interval numbers and these mathematical structures have been very popular since 1965.

Interval analysis is based upon the very simple idea of enclosing real numbers in intervals and real vectors in boxes. This ideology makes it possible to obtain guaranteed results on computers by direct transposition to interval variables of classical numerical algorithms usually operating on floating point numbers.Interval arithmetic is a tool in numerical computing where the rules for the arithmetic of intervals are explicitly stated. It was first suggested by P.S.Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by R.E.Moore [8],[9] in 1959 and 1962.

A set consisting of a closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. We denote the set of all real valued closed intervals by $I\Re$. Any element of $I\Re$ may be called closed interval and denoted by $\tilde{x}$. That is, $\tilde{x} = [x_l, x_r] = \{x \in \Re : x_l \le x \le x_r\}$. An interval number $\tilde{x}$ is a closed subset of real numbers. Let x_l and x_r be respectively referred to as the infimum (lower bound) and supremum (upper bound) of the interval number $\tilde{x}$. If $\tilde{x} = [0,0]$, then $\tilde{x}$ is said to be a zero interval. It is denoted by $\tilde{0}$.

For $\tilde{x}_1, \tilde{x}_2 \in I\Re$, we define $\tilde{x}_1 = \tilde{x}_2$ if and only if $x_{1l} = x_{2l}$ *and* $x_{1r} = x_{2r}$

$$\tilde{x}_1 + \tilde{x}_2 = \{x \in \Re : x_{1l} + x_{2l} \le x \le x_{1r} + x_{2r})\}$$

$$\tilde{x}_1 \times \tilde{x}_2 = \{x \in \Re : \min(x_{1l}x_{2l}, x_{1l}x_{2r}, x_{1r}x_{2l}, x_{1r}x_{2r}) \le x \le \max(x_{1l}x_{2l}, x_{1l}x_{2r}, x_{1r}x_{2l}, x_{1r}x_{2r})\}$$

The set of all interval numbers $I\Re$ is a complete metric space defined by

$$d(\tilde{x}_1, \tilde{x}_2) = \max\{|\bar{x}_{1l} - \bar{x}_{2l}|, |\bar{x}_{1r} - \bar{x}_{2r}|\}$$

In the special case $\tilde{x}_1 = [a,a]$ and $\tilde{x}_2 = [b,b]$, we obtain usual metric of $\Re$.

Let us define transformation f:N $\to \Re$, k $\to$ f(k)= $\tilde{x}_k$, then $\tilde{x} = (\tilde{x}_k)$ is called sequence of interval numbers. $\tilde{x}_k$ is called kth term of sequence $\tilde{x} = (\tilde{x}_k)$, ω^i denotes the set of all interval numbers with real terms.

A sequence $\tilde{x} = (\tilde{x}_k)$ of interval numbers is said to be convergent to the interval number $\tilde{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer k_0 such that $d(\tilde{x}_k, \tilde{x}_0) < \varepsilon$ for all $k \ge k_0$ and we denote it by $\lim_k \tilde{x}_k = \tilde{x}_0$. Equivalently $\lim_k \tilde{x}_k = \tilde{x}_0$ iff $\lim_k x_{kl} = x_{0l}$ *and* $\lim_k x_{kr} = x_{0r}$.

II MAIN RESULTS

Definition 2.1.

Let $\tilde{\lambda} = (\tilde{\lambda}_k)$ be a positive sequence of interval numbers such that $\tilde{\lambda}_k \neq \tilde{0}$ for all k. We define the subspace of interval numbers $\Gamma_\pi(IR)$ as $\Gamma_{\lambda,\pi}(IR)$. $\Gamma_{\lambda,\pi}(IR)$ represent the set of all sequences of interval numbers $(\tilde{x}_k) \in \Gamma_\pi(IR)$ such that $\tilde{\lambda}\,\tilde{x} \in \Gamma_\pi(IR)$.

That is, $\Gamma_{\lambda,\pi}(IR) = \{(\tilde{x}_k) \in \Gamma_\pi(IR) : \tilde{\lambda}\,\tilde{x} \in \Gamma_\pi(IR)\}$

Theorem 2.2.

$\Gamma_{\lambda,\pi}(IR) = \Gamma_\pi(IR)$ if and only if $\limsup_k D(\tilde{\lambda}_k, \tilde{0}) < \infty$

Proof.

Suppose $\Gamma_{\lambda,\pi}(IR) = \Gamma_\pi(IR)$. Let $\frac{\tilde{x}}{\pi} = (\tilde{x}_k) \in \Gamma_{\lambda,\pi}(IR)$ then $\tilde{\lambda}\,\tilde{x} \in \Gamma_\pi(IR)$. Also for given $\varepsilon > 0$, there exists $n_0 \in N$ such that $D\left(\tilde{\lambda}_k \frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) < \varepsilon$ for all $k \geq n_0$. Suppose $\limsup D(\tilde{\lambda}_k, \tilde{0}) = \infty$, then there exists a subsequence $\{n_k\}$ such that $D(\tilde{\lambda}_{n_k}, \tilde{0}) > M$ for some $M > 0$. Therefore $\max\{|\underline{\lambda}_{n_k}|^{1/k}, |\overline{\lambda}_{n_k}|^{1/k}\} > M$. This implies that $|\underline{\lambda}_{n_k}|^{1/k} > M, |\overline{\lambda}_{n_k}|^{1/k}\} > M$.

Now , we define a sequence $\left(\frac{\tilde{x}_{n_k}}{\pi_{n_k}}\right)$ of interval numbers by

$$\frac{\tilde{x}_{n_k}}{\pi_{n_k}} = \begin{cases} [1,1] & if \quad n = k \\ [0,0] & if \quad n \neq k \end{cases}$$

Note that $(\tilde{x}_{n_k}) \in \Gamma(IR)$

but $|\underline{x}_{n_k}\,\underline{\lambda}_{n_k}|^{1/k} > M$, $|\underline{x}_{n_k}\,\overline{\lambda}_{n_k}|^{1/k} > M$, $|\overline{x}_{n_k}\,\underline{\lambda}_{n_k}|^{1/k} > M$ and $|\overline{x}_{n_k}\,\overline{\lambda}_{n_k}|^{1/k} > M$,which contradicts the fact that $D\left(\tilde{\lambda}_k \frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) < \varepsilon$. Therefore $\limsup D(\tilde{\lambda}_k, \tilde{0}) < \infty$.

Conversely,suppose $\limsup D(\tilde{\lambda}_k, \tilde{0}) < \infty$ .Then there exists $M > 0$ such that $D(\tilde{\lambda}_{n_k}, \tilde{0}) < M$ for all k.

$$\Gamma_{\lambda,\pi}(IR) \supseteq \Gamma_\pi(IR) \tag{2.1}$$

Let $\tilde{x} \in \Gamma_\pi(IR)$,then $D\left(\frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) < \frac{\varepsilon}{M}$

Now,

$$D\left(\tilde{\lambda}_k \frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) \leq D(\tilde{\lambda}_k, \tilde{0})\, D\left(\frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right)$$

$$< M.\frac{\varepsilon}{M}$$

$$= \varepsilon$$

Hence $D\left(\tilde{\lambda}_k \frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) < \varepsilon$. Therefore $\tilde{\lambda}\,\tilde{x} \in \Gamma_\pi(IR)$. Hence $\frac{\tilde{x}}{\pi} = (\tilde{x}_k) \in \Gamma_{\lambda,\pi}(IR)$.

This implies that $\Gamma_\pi(IR) \subseteq \Gamma_{\lambda,\pi}(IR)$ (2.2)

From (2.1) and (2.2), $\Gamma_{\lambda,\pi}(IR) = \Gamma_\pi(IR)$ .This completes the proof.

Theorem 2.3.

If $\tilde{\lambda} = (\tilde{\lambda}_k)$ and $\tilde{\mu} = (\tilde{\mu}_k)$ are any two fixed sequences of Interval numbers and if $D(\tilde{\gamma}_k, \tilde{0}) < M$ for all k and for some $M > 0$ where $\tilde{\gamma}_k = \frac{\tilde{\mu}_k}{\tilde{\lambda}_k}$ then $\Gamma_{\lambda,\pi}(IR) \subset \Gamma_{\mu,\pi}(IR)$.

Proof.

Suppose $D(\tilde{\gamma}_k, \tilde{0}) < M$ for some $M > 0$. Then $\sup_k \max\left\{ \left|\underline{\gamma}_k\right|^{1/k}, \left|\overline{\gamma}_k\right|^{1/k} \right\} < M$.

This implies that $\left|\frac{\underline{\mu}_k}{\underline{\lambda}_k}\right|^{1/k} < M$ and $\left|\frac{\overline{\mu}_k}{\overline{\lambda}_k}\right|^{1/k} < M$

From this, we get $\left|\underline{\mu}_k\right| < M^k \left|\underline{\lambda}_k\right|$ and $\left|\overline{\mu}_k\right| < M^k \left|\overline{\lambda}_k\right|$ (2.3)

Let $\frac{\tilde{x}}{\pi} = (\tilde{x}_k) \in \Gamma_{\lambda,\pi}(IR)$. Then $\tilde{\lambda}\ \tilde{x} \in \Gamma(IR)$. Therefore for given $\varepsilon > 0$, there exists $n_0 \in N$ such that $D\left(\tilde{\lambda}_k \frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) < \frac{\varepsilon}{M}$ for all $k \geq n_0$

Hence $\sup_k \max\left\{ \left|\underline{\lambda}_k \underline{x}_k\right|^{1/k}, \left|\underline{\lambda}_k \overline{x}_k\right|^{1/k}, \left|\overline{\lambda}_k \underline{x}_k\right|^{1/k}, \left|\overline{\lambda}_k \overline{x}_k\right|^{1/k} \right\} < \frac{\varepsilon}{M}$ (2.4)

Using (2.3) and (2.4), we get

$$\begin{aligned} \left|\underline{\mu}_k \underline{x}_k\right|^{1/k} &< M \left|\underline{\lambda}_k \underline{x}_k\right|^{1/k} \\ &< M \frac{\varepsilon}{M} \\ &< \varepsilon \end{aligned}$$

Similarly,

$$\begin{aligned} \left|\underline{\mu}_k \overline{x}_k\right|^{1/k} &< \varepsilon, \left|\overline{\mu}_k \underline{x}_k\right|^{1/k} \\ &< \varepsilon, \left|\overline{\mu}_k \overline{x}_k\right|^{1/k} \\ &< \varepsilon \end{aligned}$$

Hence $\sup_k \max\left\{ \left|\underline{\mu}_k \underline{x}_k\right|^{1/k}, \left|\underline{\mu}_k \overline{x}_k\right|^{1/k}, \left|\overline{\mu}_k \underline{x}_k\right|^{1/k}, \left|\overline{\mu}_k \overline{x}_k\right|^{1/k} \right\} < \varepsilon$

This implies that $D\left(\tilde{\mu} \frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) < \varepsilon$. Therefore $\frac{\tilde{x}}{\pi} = (\tilde{x}_k) \in \Gamma_{\mu,\pi}(IR)$. Hence $\Gamma_{\lambda,\pi}(IR) \subset \Gamma_{\mu,\pi}(IR)$.

Remark 1.

The condition stated in theorem 2.3 is not necessary. Let us define a sequence of interval numbers $(\tilde{\lambda}_k)$ as follows $(\tilde{\lambda}_k) = \left(\left[\frac{1}{k!}, \frac{1}{k!}\right]\right)$, where $k \in N$ and $(\tilde{\mu}_k) = ([1,1])$ for all k. Then $D(\tilde{\lambda}_k, \tilde{0})$ and $D(\tilde{\mu}_k, \tilde{0})$ are bounded sequences.

Thus by theorem 2.2. $\Gamma_{\lambda,\pi}(IR) = \Gamma_{\mu,\pi}(IR) = \Gamma_{\pi}(IR)$. Therefore, $\Gamma_{\lambda,\pi}(IR) \subseteq \Gamma_{\mu,\pi}(IR)$ But $D\left(\frac{\tilde{\mu}_k}{\tilde{\lambda}_k}, \tilde{0}\right) < M$ is unbounded.

Remark 2.

The metric of $\Gamma_{\lambda,\pi}(IR)$ is given by

$$d_{\lambda,\pi}\left(\frac{\tilde{x}_k}{\pi_k}, \frac{\tilde{y}_k}{\pi_k}\right) = \sup\max\left\{ \left|\underline{\lambda}_k\right|^{\frac{1}{k}} \left|\frac{\underline{x}_k - \underline{y}_k}{\pi_k}\right|^{\frac{1}{k}}, \left|\underline{\lambda}_k\right|^{\frac{1}{k}} \left|\frac{\overline{x}_k - \overline{y}_k}{\pi_k}\right|^{\frac{1}{k}} \right\}$$

And $\Gamma_{\lambda,\pi}(IR)$ is a metric space with respect to $d_{\lambda,\pi}$.

Now $\Gamma_{\lambda,\pi}(IR)$ is endowed with two topologies. One is the metric topology τ_π given by the metric $\tilde{d}$. The other topology $\tau_{\lambda,\pi}$ whose metric is $d_{\lambda,\pi}$.

Theorem 2.4.

If $\limsup D(\tilde{\lambda}_k,\tilde{0})<\infty$ then τ_π is finer than $\tau_{\lambda,\pi}$.

Proof.

It is enough to prove if $\left(\frac{\tilde{x}_k}{\pi_k}\right)$ is a sequence of interval numbers converging to $\frac{\tilde{x}}{\pi}$ in $(\Gamma_{\lambda,\pi}(IR),\tau_\pi)$, then the sequence of interval numbers converging to $\frac{\tilde{x}}{\pi}$ in $(\Gamma_{\lambda,\pi}(IR),\tau_{\lambda,\pi})$.

Consider the identity mapping I from $(\Gamma_{\lambda,\pi}(IR),\tau)$ to $(\Gamma_{\lambda,\pi}(IR),\tau_{\lambda,\pi})$. Take $\frac{\tilde{x}}{\pi}=\tilde{0}$ where $\tilde{0}$ is the zero element of $\Gamma_\pi(IR)$.

Since $\left(\frac{\tilde{x}_k}{\pi_k}\right)$ converging to $\frac{\tilde{x}}{\pi}=\tilde{0}$ in $(\Gamma_{\lambda,\pi}(IR),\tau)$, for a given $\varepsilon>0$, there exists a positive integer $n_0\in N$ such that $D\left(\frac{\tilde{x}_k}{\pi_k},\tilde{0}\right)<\varepsilon$, $(\tilde{x}_k)\in\Gamma_{\lambda,\pi}(IR)$

Then $d_{\lambda,\pi}\left(\frac{\tilde{x}_k}{\pi_k},\frac{\tilde{y}_k}{\pi_k}\right)=\sup\max\{|\underline{\lambda}_k|^{1/k}|\underline{x}_k|^{1/k},|\overline{\lambda}_k|^{1/k}|\overline{x}_k|^{1/k}\}\le U\sup\max\{|\underline{x}_k|^{1/k},|\overline{x}_k|^{1/k}\}$

where $U=\max\{|\underline{\lambda}_k|^{1/k},|\overline{\lambda}_k|^{1/k}\}$ since $\limsup D(\tilde{\lambda}_k,\tilde{0})<\infty$

Therefore, $\sup\max\{|\underline{\lambda}_k|^{1/k}|\underline{x}_k|^{1/k},|\overline{\lambda}_k|^{1/k}|\overline{x}_k|^{1/k}\}\le U\sup_k D\left(\frac{\tilde{x}_k}{\tilde{\pi}_k},\tilde{0}\right)$

This implies that $d_{\lambda,\pi}\left(\frac{\tilde{x}_k}{\pi_k},\tilde{0}\right)\le U\varepsilon$

Hence $\left(\frac{\tilde{x}_k}{\pi_k}\right)$ converging to $\frac{\tilde{x}}{\pi}=\tilde{0}$ in $(\Gamma_{\lambda,\pi}(IR),\tau_{\lambda,\pi})$. Hence τ_π is finer than $\tau_{\lambda,\pi}$.

Theorem 2.5.

$(\Gamma_{\lambda,\pi}(IR),\tau_\lambda)$ is a complete metric space if $\liminf D(\tilde{\lambda}_k,\tilde{0})>0$.

Proof.

Let $\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}}\right)$ be a interval number fundamental sequence in $\Gamma_{\lambda,\pi}(IR)$. Then for a given $\varepsilon>0$ there exists $n_0\in N$ such that

$$d_{\lambda,\pi}\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}},\frac{\tilde{x}^{(m)}}{\pi^{(m)}}\right)<\varepsilon \text{ for all } n,m\ge n_0$$

For each k, we have

$$d_{\lambda,\pi}(\overline{x}_k^{(n)},\overline{x}_k^{(m)})<\varepsilon \text{ for all } n,m\ge n_0$$

$$\max\left\{|\underline{\lambda}_k|^{\frac{1}{k}}\left|\frac{\underline{x}_k^{(n)}-\underline{x}_k^{(m)}}{\pi_k}\right|^{\frac{1}{k}},|\overline{\lambda}_k|^{\frac{1}{k}}\left|\frac{\overline{x}_k^{(n)}-\overline{x}_k^{(m)}}{\pi_k}\right|^{\frac{1}{k}}\right\}<\in \quad \text{for all } n,m\ge n_0 \qquad (2.5)$$

Let $L=\liminf D(\tilde{\lambda}_k,\tilde{0})>0$

$$L = \liminf \max\{|\underline{\lambda}_k|^{1/k}, |\overline{\lambda}_k|^{1/k}\} \quad (2.6)$$

By using (2.5) and (2.6), we get

$$\left|\frac{\underline{x}_k^{(n)} - \underline{x}_k^{(m)}}{\pi_k}\right|^{\frac{1}{k}} < \frac{\in}{L}, \left|\frac{\overline{x}_k^{(n)} - \overline{x}_k^{(m)}}{\pi_k}\right|^{\frac{1}{k}} < \frac{\in}{L}$$

Hence $\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}}\right)$ is a interval number fundamental sequence in IR and since IR is complete,

$$\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}}\right) \to \frac{\tilde{x}_k}{\pi_k} \, as \, n \to \infty \quad (2.7)$$

Hence $D\left(\frac{\tilde{x}_k^{(n)}}{\pi_k^{(n)}}, \frac{\tilde{x}_k}{\pi_k}\right) < \frac{\varepsilon}{L}$

$$\left|\frac{\underline{x}_k^{(n)} - \underline{x}_k}{\pi_k}\right|^{\frac{1}{k}} < \varepsilon/L, \left|\frac{\overline{x}_k^{(n)} - \overline{x}_k}{\pi_k}\right|^{\frac{1}{k}} < \varepsilon/L$$

Also, $|\underline{\lambda}_k|^{1/k}\left|\frac{\underline{x}_k^{(n)} - \underline{x}_k}{\pi_k}\right|^{\frac{1}{k}} < \varepsilon, \; |\overline{\lambda}_k|^{1/k}\left|\frac{\overline{x}_k^{(n)} - \overline{x}_k}{\pi_k}\right| < \varepsilon$

Hence $\sup \max\left\{|\underline{\lambda}_k|^{\frac{1}{k}}\left|\frac{\underline{x}_k^{(n)} - \underline{x}_k}{\pi_k}\right|^{\frac{1}{k}}, |\overline{\lambda}_k|^{\frac{1}{k}}\left|\frac{\overline{x}_k^{(n)} - \overline{x}_k}{\pi_k}\right|^{\frac{1}{k}}\right\} < \varepsilon$

$$\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}}\right) \to \frac{\tilde{x}_k}{\pi_k} \, as \, n \to \infty \text{ in } \Gamma_{\lambda,\pi}(IR), d_{\lambda,\pi}$$

since each $\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}}\right)$ is in $\Gamma(IR)$, we have $D\left(\frac{\tilde{x}_k^{(n)}}{\pi_k^{(n)}}, \tilde{0}\right) < \varepsilon/L$ (2.8)

By using (2.7) and (2.8), we get

$$D\left(\tilde{\lambda}_k \frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right) \le D(\tilde{\lambda}_k, \tilde{0}) \, D\left(\frac{\tilde{x}_k}{\pi_k}, \tilde{0}\right)$$

$$\le D(\tilde{\lambda}_k, \tilde{0})\left[D\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}}, \frac{\tilde{x}_k}{\tilde{\pi}_k}\right) + D\left(\frac{\tilde{x}^{(n)}}{\pi^{(n)}}, \tilde{0}\right)\right]$$

$$< L\left(\frac{\varepsilon}{L} + \frac{\varepsilon}{L}\right)$$

$$< \varepsilon$$

Hence $(\tilde{x}_k) \in \Gamma_\pi(IR)$ and so $\Gamma_{\lambda,\pi}(IR)$ is complete.

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