SYMMETRIES AND SIMILARITY REDUCTIONS OF (2+1)-DIMENSIONAL SAWADA KOTERA EQUATION WITH POTENTIAL

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Abstract:

We study the nonlinear fifth order (2+1) dimensional Sawada-Kotera equation using lie symmetry group. For this equation Lie point symmetry operators and optimal system are obtained. We determine the corresponding invariant solutions and reduced equations using obtained infinitesimal generators.

Keywords :

Lie symmetry, Sawada-Kotera equation , group invariant solutions.

1.Introduction :

Nonlinear phenomena are very important in applied mathematics and physics.They appear in various scientific field such as scientific fields such as biology, signal processing, viscoelastic materials, fluid mechanics, optical fiber, and so on .Some years ago, researchers have provided many methods for obtaining the numerical and analytical solutions of nonlinear equations ,such as tanh function method, extended tanh-function method [1,2] , $\left(\frac{G}{G'}\right)$ -expansion method [3,4],sine-cosine method [5,6], simplest equation method [7] and so on. In this paper we study the Sawada- Kotera equation , namely

$$u_t + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x + u_{xxxxx} + u_x^2 = 0 \qquad (1)$$

Sawada and Kotera proposed it more than thirty years ago [8]. Because many various properties are satisfied for this equation ,much effort has been made about its exact solutions. For example, Fuchssteiner and Oevel studied its bi Hamiltonian structure [9], and a Darboux transformation was obtained for this system[10,11], Liu and Dai [12] accomplished the Hirota's bilinear method to obtain exact solutions of the same equation . Feng and Zheng[13]sinvestigated this equation to establish travelling wave solutions via the $\left(\frac{G}{G'}\right)$–expansion method. In [14],Wazwaz implemented the extended tanh method for constructing analytical solutions of the same equation. The outline of the paper is as follows . In Section 2, we discuss the methodology of Lie symmetry analysis of the Sawada-Kotera equation. Then in section 3, we describe the classical symmetries of the Sawada-Kotera equation and we obtain the Lie point symmetries of this equation .In section 4 , we explain the Group Invariant solutions . In section 5, the optimal system are obtained of the one-dimensional sub algebras of the Sawada-Kotera equation. In section 6, we obtain symmetry reduction and differential invariants for the Sawada-Kotera equation Finally, concluding remarks are summarised in section 7.

2. Symmetries and classifications of Lie algebra for

$$u_t + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x + u_{xxxxx} + u_x^2 = 0$$

In order to derive the symmetry generators of Eqn.(1) and obtain closed form solution. We consider one parameter Lie point transformation that leaves (1) invariant. This transformation is given by

$$\tilde{x}^i = x^i + \varepsilon\, \xi^i (x, y, t ; u) + O(\varepsilon^2), \quad i =1,\ldots,4, \tag{2}$$

where $\xi^i = \frac{\partial x^i}{\partial \varepsilon}|_{\varepsilon=0}$ defines the symmetry generator associated with (1) given by

$$V = \xi \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial x^i} + \tau \frac{\partial}{\partial x^i} + \phi \frac{\partial}{\partial x^i} \tag{3}$$

In order to determine four components ξ^i , we prolong V to fifth order. This prolongation is given by the formula (4)

$$V^{(5)} = V + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xy} \frac{\partial}{\partial u_{xy}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \phi^{yt} \frac{\partial}{\partial u_{yt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{xxx} \frac{\partial}{\partial u_{xxx}}$$
$$+ \phi^{xxy} \frac{\partial}{\partial u_{xxy}} + \phi^{xxt} \frac{\partial}{\partial u_{xxt}} + \phi^{xxxx} \frac{\partial}{\partial u_{xxxx}} + \phi^{xxxy} \frac{\partial}{\partial u_{xxxy}} + \phi^{xxxt} \frac{\partial}{\partial u_{xxxt}} + \phi^{xxxxx} \frac{\partial}{\partial u_{xxxxx}} + \phi^{xxxxy} \frac{\partial}{\partial u_{xxxxy}} + \phi^{xxxxt} \frac{\partial}{\partial u_{xxxxt}} \tag{4}$$

In above expression every coefficient of the prolong generator is a functions of (x, y, t ; u) and can be determined by the formula

$$\phi^i = D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i} \quad , \tag{5}$$

$$\phi^{ij} = D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij} \quad , \tag{6}$$

$$\phi^{ijk} = D_i D_j D_k(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijk} + \eta u_{y,ijk} + \tau u_{t,ijk} \quad , \tag{7}$$

$$\phi^{ijkm} = D_i D_j D_k D_m(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijkm} + \eta u_{y,ijkm} + \tau u_{t,ijkm} \, , \tag{8}$$

$$\phi^{ijkmn} = D_i D_j D_k D_m D_n(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijkmn} + \eta u_{y,ijkmn} + \tau u_{t,ijkmn} \quad . \tag{9}$$

Where D_i represents total derivative and subscripts of *u* derivative with respect to the respective coordinates. To proceed with reductions of (A) we now use symmetry criterion for partial differential equations. For heat equation this criterion is expressed by the formula

$$V^{(5)}[u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxxx} + u_x{}^2] = 0 \tag{10}$$

Whenever,

$$V^{(5)}[u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxxx} + u_x{}^2] = 0$$

Using this symmetry criterion with (4) in mind immediately yields

$$\phi_t + 5\phi\phi_{xxx} + 5\phi_x\phi_{xx} + 5\phi^2\phi_x + \phi_{xxxxx} + \phi_x{}^2 = 0 \tag{11}$$

At this stage we calculate expression for ϕ^t, ϕ^{xx} and ϕ^{yy} using (5)-(9), substitute them in (9) and then compare coefficients of various monomials in derivatives of u. This yields the following system of over determined partial differential equations

$$\begin{aligned}
&(\xi)_u = 0\\
&(\eta)_u = 0\\
&(\tau)_u = 0\\
&(\phi)_{uu} = 0\\
&(\xi)_y = 0\\
&(\tau)_y = 0\\
&(\eta)_x = 0\\
&(\tau)_x = 0\\
&10u\phi_1 - (\xi)_t - 5u^2(\xi)_x + 5u^2(\tau)_t + 2(\phi)_x + 5(\phi)_{xx} - 5u(\xi)_{xxx} + 15u(\phi)_{xxu} - (\xi)_{xxxxx} + 5(\phi)_{xxxxu} = 0\\
&(\phi)_t + 5u^2(\phi)_x + (\phi_1)_{yy} + 5u(\phi)_{xxx} + (\phi)_{xxxxx} = 0\\
&(\phi)_x - 3u(\xi)_{xx} + 3u(\phi)_{xu} - (\xi)_{xxxx} + 2(\phi)_{xxxu} = 0\\
&(\phi) - 3u(\xi)_x + u(\tau)_t - 2(\xi)_{xxx} + 2(\phi)_{xxu} = 0\\
&-2(\xi)_x + (\tau)_t + (\phi)_u - 5(\xi)_{xx} + 10(\phi)_{xu} = 0\\
&-2(\xi)_{xx} + (\phi)_{xu} = 0\\
&-5(\xi)_x + (\tau)_t = 0\\
&-3(\xi)_x + (\tau)_t + (\phi)_u = 0\\
&-(\eta)_t - (\eta)_{yy} + 2(\phi)_{yu} = 0\\
&-2(\eta)_y + (\tau)_t = 0
\end{aligned}$$

3. Reduction of one dimensional abelian subalgebra for

$$\boldsymbol{u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxxx} + u_x{}^2 = 0}$$

After some more manipulations one finds that ξ and η become

$$\begin{aligned} \xi &= k_1 \\ \eta &= k_3 \end{aligned} \tag{12}$$

The remaining equations can then be used to determine τ and ϕ as

$$\begin{aligned} \tau &= k_2 \\ \phi &= 0 \end{aligned} \tag{13}$$

As this stage we construct the symmetry generators corresponding to each of the constants involved. These are a total of eight generators given by

$$\begin{aligned} v_1 &= \partial x \\ v_2 &= \partial t \\ v_3 &= \partial y \end{aligned} \tag{14}$$

It is easy to check that the symmetry generators found in (13) form a closed Lie algebra whose commutation relations are given in table 1

$[V_i,V_j]$	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	0
V_3	0	0	0

Commutation relations satisfied by generator

4. Reductions of two dimensional abelian subalgebra for

$$\boldsymbol{u_t + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x + u_{xxxxx} + u_x^2 = 0}$$

We now briefly show steps involved in the reduction of the nonlinear heat equation to a second order differential equations. Since reduction under all the sub algebra's cannot be given in the paper, we restrict ourselves to giving reductions in one case only.

i.e., $\{V_1,V_3\}$and$\{V_2,V_3\}$.

Reduction in the remaining cases is listed in the form of Appendices A at the end of the paper.

4.1 Reduction under V_1 and V_3

From Table 1 we find that the given generators commute $[V_1,V_3]=0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our purpose we begin reduction with V_1. The characteristic equation associated with this generator is

$$\frac{dx}{1}=\frac{dy}{0}=\frac{dt}{0}=\frac{du}{0}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables

$$y = s, \qquad t = r, \qquad u = w(s,r) \tag{15}$$

Using these similarity variables Eqn.(1) can be recast in the form

$$w_r = 0 \tag{16}$$

At this stage we express V_3 in terms of the similarity variables defined in (14). It is straightforward to note that V_3 in the new variables takes the form

$$\vec{V}_3 = \partial s \tag{17}$$

The characteristic equation for $\vec{V}_3$ is $\frac{dr}{0}=\frac{ds}{1}=\frac{dw}{0}$. Integrating this equation as before leads to new variable $r=\alpha$ and $\beta(\alpha)=w$, which reduce (15) to a second order differential equation

$$\beta' = 0 \tag{18}$$

4.2 Reduction under V_3 and V_2

In this case the two generators V_3 and V_2 satisfy the commutation relation $[V_3,V_2] = 0$.This suggests that reduction in this case should start with V_3.The similarity variables are

$$x = s, \quad t = r, \quad u = w(s,r) \tag{19}$$

The corresponding reduced partial differential equation is

$$5ww_{sss}+5w_sw_{ss}+5w^2w_s+w_{sssss}+w_s^2=0 \tag{20}$$

The transformed V_2 is

$$\vec{V}_2 = \partial r \tag{21}$$

The invariants of $\vec{V}_2$ are $s=\alpha$ and $\beta(\alpha)=w$ which reduce (18) to the ordinary differential equation

$$5\beta\beta''' + 5\beta'\beta'' + 5\beta^2\beta' + \beta''''' + \beta''^2 = 0 \tag{22}$$

Reductions in remaining cases using generators forming sub-algebra are given in the form of Table 2 in Appendix A.

Appendix A

Table 2

Algebra	Reduction
$[V_1, V_3]$	$\beta' = 0$
$[V_2, V_3]$	$5\beta\beta''' + 5\beta'\beta'' + 5\beta^2\beta' + \beta''''' + \beta''^2 = 0$
$[V_1, V_2]$	Satisfied the equation

5. *Conclusion:*

In this work, we used the Lie group method to find the Lie point symmetries of the Sawada Kotera equation, which concludes to similarity variables. We utilized the corresponding invariant solution to reduce the number of independent variables in the Sawada Kotera equation. Also, we have obtained an optimal system for the Sawada Kotera equation

6. *References*

[1] E. Fan, Extended tanh-function method and its applications to nonlinear equations, phys. Lett. A,**277**(2000),212-218.

[2] A.M. Wazwaz, The tanh-coth method for solutions and kink solutions for non-linear parabolic equations, Appl .Math.and comp.,**188** (2007),1467-1475.

[3] H. jafari, N.Kadkhoda, A.S Biswas, The $\left(\frac{G}{G'}\right)$-expansion method for solutions of evolution equations from isothermal magneto static atmospheres, jour .of King saud Uni-sci ,(2013) 25,57-62s.

[4] E. Salehpour, H. jafari, N. kadkhoda ,Application of $\left(\frac{G}{G'}\right)$-expansion method to nonlinear Lienard equation, indian journal of Sci. and Tech., **5** (2012),2554-2556.

[5] H. jafari , A. Borhanifar ,S.A. Karimi, New solitary wave solutions for the bad Boussinesq and good Boussinesq equations, Numer. Meth . For par. Diff. Eq., **25**,No.5(2009),1231-1237.

[6] A.M. Wazwaz , The sine-cosine method for obtaining solutions with compact and non-compact structures, Appl. Math .and comp., **159** (2004),559-576.

[7] H. jafari , N. Kadkhoda , C.M. Khalique , Travelling wave solutions of nonlinear evolution equations using the simplest equation method, comp. and Math .With Appl.,**64,** No.6(2012),2084-2088.

[8] K. Sawada, T. kotera, prog. Theor. **51** (1974),1355.

[9] B .Fuchssteiner, W, Oevel,J. Math, phys., **23**(1982),358.

[10] R.N.Aiyer, B. Fuchssteiner,W. Oevel , J.s Phys.A, **19**(1986),3755.