On the genus of order difference interval graph of a finite abelian group

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Abstract

The order difference interval graph of a group G, denoted by $\Gamma_{ODI}(G)$, is a graph with $V(\Gamma_{ODI}(G) = G$ and two vertices a and b are adjacent in $\Gamma_{ODI}(G)$ if and only if $o(a) - o(b) \epsilon [o(a), o(b)]$. Without loss of generality, assume that $o(a) \leq o(b)$. In this paper, we try to classify all finite abelian groups whose order difference interval graphs are toroidal and projective.

Keywords: order difference interval graph, finite group, planar, genus, crosscap.

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1 Introduction

There are different ways to associate to a group a certain graph. In this context, it is interesting to ask for the relation between the structure of the group, given in group theoretical terms, and the structure of the graph, given in the language of graph theory. There are many papers on assigning a graph to a group and investigating algebraic properties of group using the associated graph, for instance, see [1, 2, 3].

Let G be a finite group. One can associate a graph to G in many different ways. Since the order of an element is one of the most basic concepts of group theory, Balakrishnan and Kala [4] defined the *order difference interval graph* of a group G denoted by $\Gamma_{ODI}(G)$ as follows: Take $V(\Gamma_{ODI}(G) = G$ and two vertices a and b are adjacent in $\Gamma_{ODI}(G)$ if and only if $o(a) - o(b) \epsilon [o(a), o(b)]$. Without loss of generality, assume that $o(a) \leq o(b)$. Here $o(a)$ and $o(b)$ denote the orders of a and b, respectively. In this paper, we try to classify all finite abeliean group G whose order difference interval graph has genus at most one.

It is well known that any compact surface is either homeomorphic to a sphere, or to a connected sum of g tori, or to a connected sum of k projective planes (see [8, Theorem 5.1]). We denote by $\mathbb{S}_g$ the surface formed by a connected sum of g tori, and by $\mathbb{N}_k$ the one formed by a connected sum of k projective planes. The number g is called the genus of the surface $\mathbb{S}_g$ and k is called the crosscap of $\mathbb{N}_k$. When considering the orientability, the surfaces$\mathbb{S}_g$ and sphere are among the orientable class and the surfaces $\mathbb{N}_k$ are among the non-orientable one.

A simple graph which can be embedded in $\mathbb{S}_g$ but not in $\mathbb{S}_{g-1}$ is called a graph of *genus* g. Similarly, if it can be embedded in $\mathbb{N}_k$ but not in $\mathbb{N}_{k-1}$, then we call it a graph of *crosscap* k. The notations (G) and (G) are denoted for the genus and crosscap of a graph G, respectively. It is easy to see that $\gamma(H) \leq \gamma(G)$ and $\bar{\gamma}(H) \leq \bar{\gamma}(G)$ for all subgraph H of G. Also a graph G is called planar if $\gamma(G) = 0$, it is called toroidal if $\gamma(G) = 1$, and it is called projective if $\bar{\gamma}(G) = 1$.

A remarkable characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is *planar* if and only if it contains no subdivision of K_5 or $K_{3,3}$. A graph is *outerplanar* if it can be embedded into the plane so that all its vertices lie on the same face.

Throughout this paper, we assume that G is a finite group. We denote the group of integers addition modulo n by $\mathbb{Z}_n$ and the Euler function by $\phi(n)$. For basic definitions on groups, one may refer [7].

2 GENUS AND CROSSCAP OF $\Gamma_{ODI}(G)$

The main goal of this section is to determine all finite abelian groups G whose order difference interval graph has genus one.

Lemma 2.1. [4] Let a be a generator element in group G of order n. Then a is adjacent to all the non-generator elements of G in the graph $\Gamma_{ODI}(G)$.

In view of preceding lemma, we have the following result.

Lemma 2.2. *Let G be a cyclic group of order n. Then $\Gamma_{ODI}(G)$ has a subgraph isomorphic to $K_{\phi(n),n-\phi(n)}$. Moreover, if n is prime, then $\Gamma_{ODI}(G) \cong K_{1,n-1}$.*

The following characterization of outerplanner graphs was given by Chartrand and Harary [6]. Using this characterization, we charcterize all finite groups G whose $\Gamma_{ODI}(G)$ is outerplanar.

Theorem 2.3. [6] A graph G is outerplanar if and only if it contains no subdivisionof K_4 or $K_{2,3}$.

Theorem 2.4. *Let G be a finite abelian group. Then $\Gamma_{ODI}(G)$ is outerplanar if and only if G is isomorphic to $\mathbb{Z}_p^n$,, $n \geq 1$ or $\mathbb{Z}_4$, where p is a prime.*

Proof. Assume that$\Gamma_{ODI}(G)$ is outerplanar. Since G is finite, $|G| = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, n \geq 1$.

Suppose G has a cyclic subgroup of order $p_1 p_2$. Then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{2,3}$ as a subgraph, a contradiction. Hence G is a p-group and so $|G| = p^n$.

Suppose G has an element of order $p^m, m \geq 3$. Then G has a cyclic subgroup H of order p^m. Then by Lemma 2.2, $\Gamma_{ODI}(H)$contains $K_{4,4}$ as a subgraph. Therefore $\Gamma_{ODI}(G)$ has a subgraph which is isomorphic to $K_{4,4}$, a contradiction. Thus G has elements of order at most p^2.

If order of every element of G is p, then $G \cong \mathbb{Z}_p^n$.

Suppose G has an element of order p^2. If $p \geq 3$, then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{2,3}$ as a subgraph, which is a contradiction. Hence p must be 2.

Suppose G is a cyclic 2-group, then $G \cong \mathbb{Z}_4$.

Suppose G is a non-cyclic 2-group. If G has a subrgoup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, then $\Gamma_{ODI}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ contains $K_{2,3}$ as a subgraph. Hence $\Gamma_{ODI}(G)$ contains $K_{2,3}$ as a subgraph, a contradiction.

Conversely, if $G \cong \mathbb{Z}_p^n$, then $\Gamma_{ODI}(G) \cong K_{1,p^n-1}$. If $G \cong \mathbb{Z}_4$, then $\Gamma_{ODI}(G) \cong K_4 - e$.

Theorem 2.5. *Let G be a finite abelian group. Then $\Gamma_{ODI}(G)$ is planar if and only if G is isomorphic to $\mathbb{Z}_p^n$, $n \geq 1$ or $\mathbb{Z}_4$, where p is a prime.*

Proof. Assume that $\Gamma_{ODI}(G)$ is a planar graph. Since G is finite, $|G| = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, n \geq 1$.

Case 1. Suppose G is a p-group. Suppose G has an element of order $p^m, m \geq 3$. Then G has a cyclic subgroup H of order p^m. Then by Lemma 2.2, $\Gamma_{ODI}(H)$ contains $K_{4,4}$ as a subgraph. Therefore $\Gamma_{ODI}(G)$ has a subgraph which is isomorphic to $K_{4,4}$, a contradiction. Thus G has elements of order at most p^2.

If every element of G is of order p, then $G \cong \mathbb{Z}_p^n$.

Suppose G has an element of order p^2. Then G has a cyclic subgroup H of order p^2. Suppose$p \geq 3$, then by Lemma 2.2, $\Gamma_{ODI}(H)$ contains a subgraph which is isomorphic to $K_{3,6}$ and so is $\Gamma_{ODI}(G)$. Hence $p = 2$.

Suppose G is a cyclic 2-group, then $G \cong \mathbb{Z}_4$.

Suppose G is a non-cyclic 2-group. If G has a subrgoup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, then $\Gamma_{ODI}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ contains $K_{3,3}$ as a subgraph, $\Gamma_{ODI}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is non-planar and so is $\Gamma_{ODI}(G)$.

Case 2. Suppose G is not a p-group. Then $|G| = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, n \geq 2$. Clearly G has a cyclic subgroup of order $m = p_1 p_2 \cdots p_n$. Consider $S_1 = \{a_i \in G : o(a_i) = m\} \cup \{e\}$and $S_2 = \{b_j \neq e \in G : o(b_j) | m \text{ and } o(b_j) \neq m\}$. It is clear that $o(b_j) \leq \frac{m}{2}$ and $o(a_i) - o(b_j) \geq \frac{m}{2}$. Hence a_i is adjacent to b_j for all i and j. Thus $\Gamma_{ODI}(G)$ contains $K_{3,3}$ as a subgraph, a contradiction.

Conversely, if $G \cong \mathbb{Z}_p^n$, then $\Gamma_{ODI}(G) \cong K_{1,p^n-1}$. If $G \cong \mathbb{Z}_4$, then $\Gamma_{ODI}(G) \cong K_4 - e$.

For a rational number q, $\lceil q \rceil$ is the first integer number greater or equal than q. In the following lemma we bring some well-known formulas for genus of a graph (see [9]).

Lemma 2.6. *The following statements hold:*

$(i)\ \gamma(K_n) = \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil \text{ if } n \geq 3$

$(ii)\ \gamma(Km,n) = \left\lceil \frac{1}{4}(m-2)(n-2) \right\rceil \text{ if } m,n \geq 2.$

In the following theorem we determine all finite groups whose $\Gamma_{ODI}(G)$ has genus one.

Theorem 2.7. *Let G be a finite abelian p-group. Then $\Gamma_{ODI}(G)$ if and only if G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_8$ or $\mathbb{Z}_9$.*

Proof. Assume that $\gamma(\Gamma_{ODI}(G)) = 1$. Since $|G| = p^n$, by Theorem 2.5, $n \geq 2$ and $G \ncong \mathbb{Z}_p^n$.

Case 1. Supppose G is cyclic. If $n \geq 4$, then by Lemma 2.2, $K_{8,8}$ is a subgraph of $\Gamma_{ODI}(G)$. Therefore by Lemma 2.6, $\gamma(\Gamma_{ODI}(G)) \geq 9$, a contradiction. Thus $n \leq 3$.

Suppose $n = 3$. If $p \geq 3$, then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{18,9}$ as a subgraph. Hence by Lemma 2.6, $\gamma(\Gamma_{ODI}(G)) \geq 28$, which is a contradiction. Hence $p = 2$ and $G \cong \mathbb{Z}_8$.

Suppose $n = 2$. If $p \geq 5$, then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{20,5}$ as a subgraph. Therefore by Lemma 2.6,$\gamma(\Gamma_{ODI}(G)) \geq 14$, a contradiction. Hence $p = 2, 3$. By Theorem 2.5, $\Gamma_{ODI}(\mathbb{Z}_4)$ is planar and hence $G \cong \mathbb{Z}_9$.

Case 2. Suppose G is non-cyclic. Then G has a subgroup which is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^m}$, $m \geq 2$. Consider the sets $S_1 = \{x \in G : |x| = p\}$, $S_2 = \{y \in G : |y| = p^m\}$. Clearly $|S_1| \geq p^2 - 1$ and $|S_2| \geq p^m(p-1)$. If $p \geq 3$, then the subgraph induced by $S_1 \cup S_2$ contains $K_{8,18}$ as a subgraph. Therefore$\gamma(\Gamma_{ODI}(G)) \geq 24$, a contradiction. Thus $p = 2$. If $m \geq 3$, then $\Gamma_{ODI}(G)$ contains $K_{3,8}$ as a subgraph and so $\gamma(\Gamma_{ODI}(G)) \geq 2$, a contradiction.

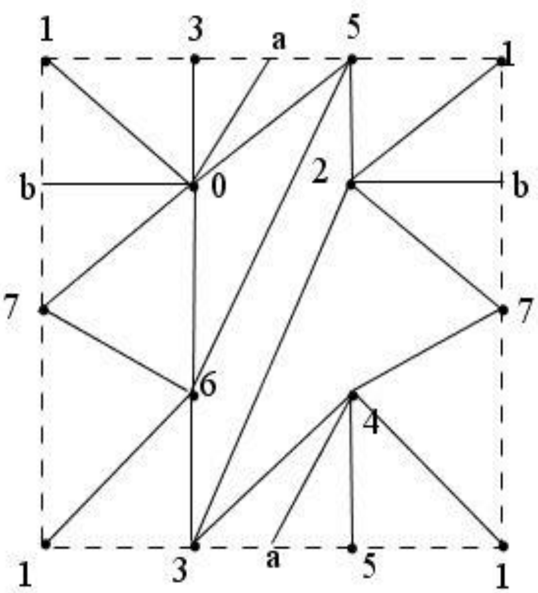

Fig. 1 Embedding of $\Gamma_{ODI}(\mathbb{Z}_8) \cong \Gamma_{ODI}(\mathbb{Z}_2 \times \mathbb{Z}_4)$

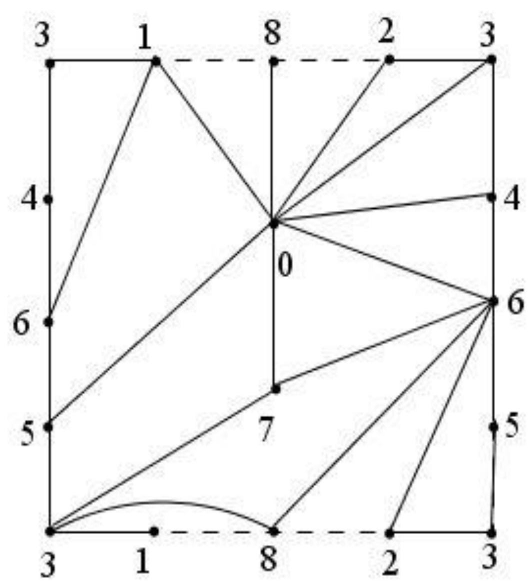

Fig. 2 Embedding of $\Gamma_{ODI}(\mathbb{Z}_9)$

Suppose G has a subgroup H which is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^2}$. Then consider $S_1 = \{a \in H : |a| = p^2\}$ and $S_2 = \{b \in H : |b| = p\}$. It is clear that $|S_1| \geq 8$ and $|S_2| \geq 7$. Hence the subgraph induced by $S1 \cup S2$ is isomorphic to $K_{8,7}$. Therefore by Lemma 2.6, $\gamma(\Gamma_{ODI}(H)) \geq 8$ and so is $\gamma(\Gamma_{ODI}(G)) \geq 8$, which is a contradiction.

Suppose G has a subgroup H which is isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$. Then it is easily seen that $K_{4,5}$ as a subgraph of $\Gamma_{ODI}(G)$. Therefore by Lemma 2.6, $\gamma(\Gamma_{ODI}(H)) \geq 2$, a contradiction. Thus $G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Converse follows from Figs. 1, 2.

Theorem 2.8. *Let G be a finite abelian non-p-group. Then $\gamma\left(\Gamma_{ODI}(G)\right) = 1$ if and only if G is isomorphic to $\mathbb{Z}_6$.*

Proof. Assume that $\gamma\left(\Gamma_{ODI}(G)\right) = 1$. Since G is not a p-group, $|G| = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, n \geq 2$.

Case 1. G is a cyclic group.

If $n \geq 3$, then by Lemma 2.2, $K_{8,22}$ as a subgraph of $\Gamma_{ODI}(G)$. Therefore by Lemma 2.6, $\gamma(\Gamma_{ODI}(H)) \geq 40$. Thus $n = 2$. Suppose $k_i \geq 2$ for some i. Then by Lemma 2.2, $K_{4,8}$ as a subgraph of $\Gamma_{ODI}(G)$. Hence by Lemma 2.6, $\gamma(\Gamma_{ODI}(G)) \geq 3$, which is a contradiction. Therefore $k_i = 1, i = 1, 2$ and so $|G| = p_1 p_2$. If $p_i \geq 5$, then by Lemma 2.2, $\Gamma_{ODI}(G)$ contains $K_{4,6}$ as a subgraph, a contradiction. Therefore $|G| = 6$ and hence $G \cong \mathbb{Z}_6$.

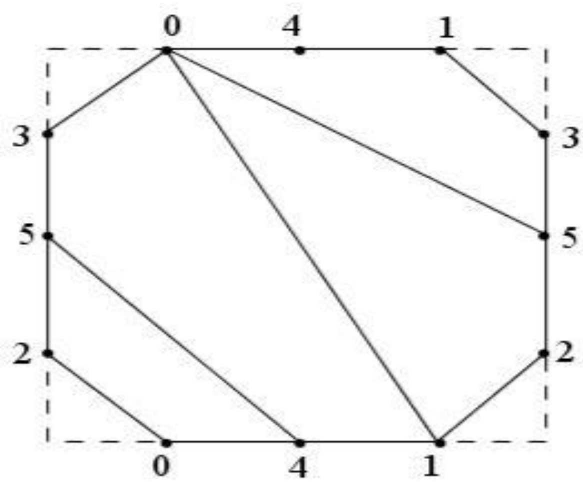

Fig. 3 Embedding of $\Gamma_{ODI}(\mathbb{Z}_6)$

Case 2. Suppose G is not a cyclic group. Then G has a subgroup H which is isomorphic to $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$. Consider $S_1 = \{x \in H : |x| = p_1 p_2\}$ and $S_2 = \{y \in H : |y| = p_1\} \cup \{e\}$. It is clear that, the subgraph induced by $S_1 \cup S_2$ contains $K_{4,6}$ as a subgraph, a ontradiction.

Converse follows from Fig. 3.

In the following lemma we bring some well-known formulas for crosscap of a graph (see [9]).

Lemma 2.9. *The following statements hold:*

$$(i)\ \bar{\gamma}(K_n) = \begin{cases} \left\lceil \frac{1}{12}(n-3)(n-4) \right\rceil & if\ n \geq 3\ and\ n \neq 7 \\ 3 & if\ n = 7 \end{cases}$$

$$(ii)\ \bar{\gamma}(Km,n) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil if\ m,n \geq 2.$$

By slight modifications in the proof of Theorem 2.7 and Theorem 2.8 with Lemma 2.9, one can prove the following theorem.

Theorem 2.10. *Let G be a finite abelian group. Then $\bar{\gamma}(\Gamma_{ODI}(G)$ =1 if and only if G is isomorphic to $\mathbb{Z}_6$.*

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