Domination in k-nil hypergraph of ideals of commutative ring

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Abstract

Let R be a commutative ring with identity and $k > 2$ a fixed integer. Let $\mathcal{N}(\mathrm{R}, \mathrm{k})$ be the set of all k-nil ideals of R. The k-nil hypergraph of ideals of R, denoted by $\mathcal{H}_k(R)$ is a hypergraph with vertex set $\mathcal{N}(\mathrm{R}, \mathrm{k})$ and for distinct ideals $\{I_1, I_2, \dots, I_k\}$ in $\mathcal{N}(\mathrm{R}, \mathrm{k})$, the set $\{I_1, I_2, \dots, I_k\}$ is an edge in $\mathcal{H}_k(R)$ if and only if $\prod_{i=1}^{k} I_i \subseteq Nil(\mathrm{R})$ and the product of ideals of no $(k-1)$ subset of $\{I_1, I_2, \dots, I_k\}$ is contained in $Nil(\mathrm{R})$. The main goal of this paper is to study the domination number of $\mathcal{H}_3(R)$.

Keywords: hypergraph, k-nil hypergraph, domination.

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1. Introduction

Mathematical study of domination in graphs began around 1960. This concept have received extensive attention by many researchers. Perhaps the fastest growing area within graph theory is the study of domination and related subset problems. Claude Berge [2] wrote a book on graph theory, in which he introduced the coefficient of external stability, which is now known as the domination number of a graph. Oystein Ore [7] introduced the terms dominating set and domination number in his book on graph theory which was published in 1962. An excellent treatment of fundamentals of domination is given in the book by Haynes et.al[3]. The concept of domination in graphs has been somewhat sparsely studied for hypergraphs. B. D. Acharya was the pioneer of domination in hypergraphs [1]. Shaveisi et al.,[6] introduced the nil-graph of ideals of R. Recently in [5], Selvakumar et al., extended the concept of nil-graph of ideals of a commutative ring R to k-nil hypergraph of ideals of a commutative ring . We state the definition here. Let R be a commutative ring with identity and $k > 2$ a fixed integer. Let $\mathcal{N}(\mathrm{R}, \mathrm{k})$ be the set of all k -nil ideals of R. The k -nil hypergraph of ideals of R, denoted by $\mathcal{H}_k(R)$ is a hypergraph with vertex set $\mathcal{N}(\mathrm{R}, \mathrm{k})$ and for distinct ideals $\{I_1, I_2, \dots, I_k\}$ in $\mathcal{N}(\mathrm{R}, \mathrm{k})$ the set $\{I_1, I_2, \dots, I_k\}$ is an edge in $\mathcal{H}_k(R)$ if and only if $\prod_{i=1}^{k} I_i \subseteq Nil(\mathrm{R})$ and the product of ideals of no $(k-1)$ subset of $\{I_1, I_2, \dots, I_k\}$ is contained in $Nil(\mathrm{R})$.

2. Preliminaries

In this section, we first recall the definition of hypergraph and domination. We also summarize some basic definitions.

A hypergraph $\mathcal{H}$ is an ordered pair $(X,\mathcal{E})$ where X is a non empty finite set and $\mathcal{E}$ is a subset of the power set 2^X of X, viz., the set of all subsets of X such that $E \in \mathcal{E} \Rightarrow E \neq \emptyset$ and $\bigcup_{E\in\mathcal{E}} E = X$.The elements of X are called vertices of $\mathcal{H}$ and those of $\mathcal{E}$ edges of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is called k -uniform if every edge e of $\mathcal{H}$ is of size k. For all terminology and notation in hypergraph theory not specifically defined here the reader can refer to C.Berge [2].

A set D $\subseteq$ X is a dominating set of $\mathcal{H}$ if for every $v \in X - \mathrm{D}$ there exists $u \in D$ such that u and v are adjacent in $\mathcal{H}$; that is, if there exists $E \in \mathcal{E}$ such that $u, v \in E$. A dominating set D is a minimal dominating set if no proper subset of D is a dominating set. The domination number (of $\mathcal{H}$), denoted $\gamma(\mathcal{H})$, is the minimum cardinality taken over all minimal dominating sets. A set I $\subseteq$ X is an independent set (of $\mathcal{H}$), if $\mathrm{N}(\mathrm{u}) \cap \mathrm{N}\{\mathrm{v}\} = \emptyset$ for all $u, v \in I$. A set I $\subseteq$ X is an independent domination set (of $\mathcal{H}$) if I is both an independent and dominating set. The minimum cardinality of an independent dominating set is called the independent domination number of $\mathcal{H}$ and is denoted by $\gamma_i(\mathcal{H})$. Let D be a minimum dominating set in a graph $G = (V, E)$. If $V - D$ contains a dominating set D of G, then D is called an inverse dominating set with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the cardinality of a smallest inverse dominating set of G. A comprehensive listing of the known domination related parameters has been produced in [2].

In this paper, we study domination in k −nil hypergraph of ideals. We first study domination, connected domination and inverse domination in k-nil hypergraph of ideals. Finally, we discuss about the independent domination number of $\mathcal{H}_3(R)$ and hyperdomination in k −nil hypergraph of ideals.

3. Domination in k –nil hypergraph of ideals

In this section we explore many domination parameters which are obtained by combining domination with another graph theoretical property and provide some examples. We begin this section with the following example .

Example 3.1. For $R = \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_2$, $\mathcal{N}(\mathrm{R}, 3) = \{x_1 = \mathbb{Z}_4 \times \mathbb{Z}_9 \times (0),\ x_2 = \mathbb{Z}_4 \times (3) \times \mathbb{Z}_2,\ x_3 = \mathbb{Z}_4 \times (0) \times \mathbb{Z}_2,\ x_4 = (0) \times \mathbb{Z}_9 \times \mathbb{Z}_2,\ x_5 = (2) \times \mathbb{Z}_9 \times \mathbb{Z}_2\}$. Then $e_1 = \{x_1, x_3, x_4\}$, $e_2 = \{x_1, x_2, x_4\}$, $e_3 = \{x_1, x_3, x_5\}$, $e_4 = \{x_1, x_2, x_5\}$. Here $D = \{x_1\}$.

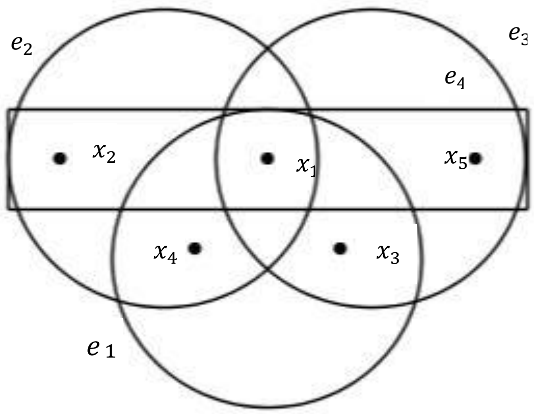

Fig. 1 $\mathcal{H}_3(\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_2)$

Throughout this paper, let $V_i = \{\prod_{n-i} R_s \times \prod_i I_t$: I_t is proper ideal in $R_t\}$ for $i = 1, 2, \ldots,$

$n - 2$. Then $\mathcal{N}(R, k) = \cup_{i=1}^{n-2} V_i$ and $V_i \cap V_j = \emptyset$ for all $i \neq j$.

Lemma 3.2. Let $R = R_1 \times \cdots \times R_n, n \geq 4$ be a ring. For each $X \in V_i, 1 \leq i \leq n-2$, there exists $Y \in \mathcal{N}(R, 3)$ such that $X, Y \in E$, for all edge E in $\mathcal{H}_3(R)$.

Proof. Let for each $1 \leq i \leq n-2$, $X_i \in V_i$. Define $\Omega = \{m | I_m \subset R_m$ in X$\}$. If $X \in V_1$, then there exists a vertex $Y = I_s \times I_t \times \prod_{n-2} R_u \in \mathcal{N}(R, 3), s \in \Omega$, such that $X, Y \notin E$ for all edge E in $\mathcal{H}_3(R)$. If $X \in V_i, 2 \leq i \leq n - 2$, then there exists a vertex $Y = \prod_{m \in \Omega^c} I_m \times \prod_{n \in \Omega} R_n \in \mathcal{N}(R, 3)$ such that $X, Y \subseteq \mathrm{Nil}(R)$, which implies $X, Y \notin E$ for all edge E in $\mathcal{H}_3(R)$.

Lemma 3.3. Let $R = R_1 \times R_2 \times R_3$, where R_1 and R_2 are local rings but not a field, R_3 is a field. Then there exists $X \in \mathcal{N}(R, 3)$ such that $X \in E$ for all edge E in $\mathcal{H}_3(R)$.

Proof. Let $R = R_1 \times R_2 \times (0) \in \mathcal{N}(R, 3)$. It is evident that X dominates all the vertices of $\mathcal{H}_3(R)$, which completes the proof.

Now we pose the following question: Which of the following types of rings have domination number one ? The following theorem answers our question.

Theorem 3.4. Let $R = R_1 \times \cdots \times R_n$ be a ring, where each (R_i, m_i) is a local ring. Then $\gamma(\mathcal{H}_3(R))=1$ if and only if $R \cong R_1 \times R_2 \times R_3$, at most two R_i is local with $m_i \neq (0)$.

Proof. Assume that $\gamma(\mathcal{H}_3(R))=1$. Then there exists a vertex $X \in \mathcal{N}(R, 3)$ such that $D = \{X\}$ is a dominating set of $\mathcal{H}_3(R)$, where $X \in V_i$ for some i. Define $\Omega = \{m | I_m \subset R_m$ in X $\}$. Suppose $n \geq 4$. By Lemma 3.2, there exists $Y \in \mathcal{N}(R, 3)$ such that $X, Y \notin E$, for all edge E in $\mathcal{H}_3(R)$. Therefore $n = 3$ and hence $R \cong R_1 \times R_2 \times R_3$ and $\mathcal{N}(R, 3) = V_1$. Suppose R_i is local with $m_i \neq (0)$ for all i. Then there exists a vertex $Y = I_s \times \prod_2 R_t \in V_1$, $s \in \Omega$ such that $X, Y \notin E$ for all edge E in $\mathcal{H}_3(R)$, a contradiction. Hence at most two R_i is local with $m_i \neq (0)$.

On the other hand, assume $R \cong R_1 \times R_2 \times R_3$, at most two R_i is local with $m_i \neq (0)$. By Lemma 3.3, there exits a vertex $X = R_1 \times R_2 \times (0) \in \mathcal{N}(\mathrm{R}, 3)$, which dominates all the vertices of $\mathcal{H}_3(R)$.

Example 3.5. Let $R = F_1 \times F_2 \times F_3 \times F_4$ be a ring, where each F_i is a field and $D = F_1 \times F_2 \times (0) \times (0), F_1 \times F_2 \times F_3 \times (0)\}$. It is easy to see that D is a dominating set and therefore $\gamma(\mathcal{H}_3(R)) = 2$.

Example 3.6. Consider Let $R = F_1 \times \cdots \times F_6$, where each F_i is a field and

$D = F_1 \times (0) \times F_3 \times (0) \times F_5 \times (0), (0) \times F_2 \times F_3 \times F_4 \times (0) \times (0), (0) \times (0) \times (0) \times F_4 \times F_5 \times F_6\}$. Then D is a dominating set and so $\gamma(\mathcal{H}_3(R)) = 3$.

In view of example 3.3 and example 3.4, if $R = \prod_{i=1}^{4} F_i$, then $\gamma\left(\mathcal{H}_3(R)\right) = 2 = n - 2$

and $\gamma\left(\mathcal{H}_3(R)\right) = 3 = n - 3$ when $R = \prod_{i=1}^{6} F_i$, the domination number of graph does not vary uniformly. Based on the value of n, the domination number of $\mathcal{H}_3(R)$ where $R = F_1 \times \cdots \times F_n$ splits into two cases.

In the following theorem, we determine domination number of 3-nil hypergraph of ideals of reduced ring with the number of maximal ideals is at least six.

Theorem 3.7. Let $R = F_1 \times \cdots \times F_n, n \geq 6$ be a ring where each F_i is a field. Then $\gamma\left(\mathcal{H}_3(R)\right) = \mathrm{n} - 3$. Moreover, $\gamma\left(\mathcal{H}_3(R)\right) = \gamma^{-1}\left(\mathcal{H}_3(R)\right) = \gamma_c\left(\mathcal{H}_3(R)\right)$.

Proof. Let $D = \{X_1, \ldots, X_{n-3}\}$, where $X_i \in V_{n-3}, 1 \leq \mathrm{i} \leq \mathrm{n} - 3$. Consider $X_l = \prod_{n-3} I_i \times \prod_3 F_j \in D$ and define $\Omega_l = \{\alpha : I_\alpha \subset F_\alpha \text{ in } X_l\}$. For each $1 \leq l \leq n - 3$, X_l dominates all the vertices of the form $\mathrm{I}_i \times \prod_{n-1} F_j \in V_1$ where $i \notin \Omega_l$, $\prod_2 \mathrm{I}_i \times \prod_{n-2} F_j \in V_2$ where $i \notin \Omega_l$, $\mathrm{I}_i \times I_s \times \prod_{n-2} F_j \in V_2$ where $i \in \Omega_l$, $\prod_{2_{i\in\Omega_l{}^c}} I_i \times I_s \times \prod_{n-3} F_j, s \in \Omega_l, I_t \times \prod_{2_{i\in\Omega_l}} I_i \times \prod_{n-3} F_j \in V_3, \prod_{2_{i\in\Omega_l}} I_i \times \prod_{2_{j\notin\Omega_l}} I_j \times \prod_{n-4} F_l \in V_4, \ldots, \prod_{n-4_{i\in\Omega_l}} I_i \times \prod_{2_{s\in\Omega_l{}^c}} I_j \times \prod_2 F_j \in V_{n-2}$. Hence D is a dominating set of $\mathcal{H}_3(R)$ and so $\gamma(\mathcal{H}_3(R)) \leq \mathrm{n} - 3$.

Suppose $S = \{Y_1, \ldots, Y_{n-4}\}$ is a dominating set of $\mathcal{H}_3(R)$. Suppose $Y_i \in \mathrm{V}_1$ for all i and define $\Omega_i = \{\alpha : I_\alpha \subset F_\alpha \text{ in } Y_i\}$. Let $\Omega = \cup_{i=1}^{4} \Omega_i$. It is clear that $|\Omega| = \mathrm{n} - 4$. Then there exists a vertex $Y = \prod_{i\in\Omega} I_i \times \prod_4 F_j$ which is not dominated by any vertex of S. Suppose $Y_i \in V_j$ for all $i, 2 \leq j \leq n - 2$ or $\mathrm{S} \cap V_j \neq \emptyset$ for at least two j. The following two cases are to be considered.

Case 1. Suppose $\cap Y_i = \{0\}$ for all $Y_i \in$ S. Then there exists a vertex $Z = \prod_{n-2} I_i \times \prod_2 F_j$ which is not dominated by any vertex of S.

Case 2. Suppose $\cap Y_i \neq \{0\}$ for all $Y_i \in$ S. Let $L = \cap Y_i$. Define $\Omega = \{\alpha : I_\alpha = F_\alpha \text{ in } L\}$. Then there exists a vertex $Y = \prod_{i\in\Omega} I_i \times \prod I_j \times \prod_{n-4} F_l$ which is not dominated by any vertex of S. We conclude, therefore, that $\gamma(\mathcal{H}_3(R)) = n - 3$. The moreover condition is clear.

Theorem 3.8. Let $R = F_1 \times \cdots \times F_n$ be a ring with identity where each F_i is a field. Then $\gamma\, \mathcal{H}_3(R)) = \mathrm{n} - 2 = \gamma{-}1(\mathcal{H}_3(R))$ if and only if n = 4 or 5.

Proof. Assume that $n = 4$ or 5. Let $D = \{X_1, \cdots, X_{n-2}\}$, where $X_i \in V_i, 1 \leq i \leq n-2$. We claim that D is a γ -set of $\mathcal{H}_3(R)$. We consider the following two cases.

Case 1. Suppose $n = 4$. Then $V = \cup_{i=1}^{2} \mathrm{V}_i$. Consider $D = \{X_1, X_2\}$, where $X_1 \in V_1$

and $X_2 \in V_2$. Define Ω_i= { m : $I_m \subset F_m$ in $X_i \in D\}$, where $i = 1, 2$. Clearly X_1 dominates all the vertices of the form $\mathrm{I}_i \times \prod_{n-1} F_l \in V_1$ and X_2 dominates all the remaining vertices of the form $\mathrm{I}_j \times \mathrm{I}_l \times \prod_{n-2} F_t \in V_2, j \in \Omega_1$. Therefore, D is a dominating set and hence γ $(\mathcal{H}_3(R)) \leq 2$.

Suppose S is a dominating set with $|S| = 1$. Consider $S = \{X\}$. If $X \in V_1$, then there exists a vertex $Y_1 = \mathrm{I}_i \times \mathrm{I}_l \times \prod_{n-2} F_j, i \in \Omega_1$ which does not form an edge with X. If $X \in V_2$, then there exists a vertex $Y_2 = \mathrm{I}_i \times \prod_{n-1} F_j, i \in \Omega_1$ which is not dominated by X.

Case 2. Suppose $n = 5$. Then $V = \cup_{i=1}^{3} \mathrm{V}_i$. Let $D = \{X_1, X_2, X_3\}$, where $X_1 \in V_1$, $X_2 \in V_2$ and $X_3 \in V_3$. Define Ω_i= { m : $I_m \subset F_m$ in $X_i \in D\}$, where $i = 1, 2, 3$. Clearly X_1 and X_2 forms an edge with all the vertices of V_1 and V_2 respectively. Also X_3 forms an edge with all the vertices of V_3 except the vertices of the form. This forms an edge with either X_1 or X_2.

Suppose S is a dominating set with $|S| \leq 2$. Without loss of generality assume that $|S| = 2$. Consider $S = \{X_1, X_2\}$. If $X_1, X_2 \in V_1$, then there exists a vertex of the form which does not form an edge with any vertex in S. Suppose $X_1, X_2 \in V_2$ or V_3 or $\mathrm{S} \cap V_i \neq \emptyset$ or $\mathrm{S} \cap V_j \neq \emptyset, i \neq j$. Proof follows from case 1 and case 2 of Theorem 3.7. We conclude, therefore, that $\gamma\ (\mathcal{H}_3(R)) = n - 2$.

Reverse inclusion follows from Theorem 3.7.

For connected domination number of $\mathcal{H}_3(R)$ we consider separately the cases when n = 4 and n = 5. The proof of the next result is dual to the above proof, we left the details here.

Theorem 3.9. Let $R = F_1 \times \cdots \times F_n$ be a ring where each F_i is a field. Then $\gamma_c(\mathcal{H}_3(R)) = n - 2$ if and only if $n = 5$.

Theorem 3.10. Let $R = F_1 \times \cdots \times F_n$ be a ring where each F_i is a field. Then $\gamma_c(\mathcal{H}_3(R)) = n - 1$ if and only if $n = 4$.

Proof. Suppose $n = 4$. Then $V = \cup_{j=1}^{2} \mathrm{V}_j$. Consider D = $\{X_1, X_2, X_3\}$ where $X_i \in V_1$ for all i. Notice that X_i dominates all vertices of V_1 and V_2. Hence it is a connected dominating set with $\gamma_c(\mathcal{H}_3(R)) \leq 3$.

Suppose S is a connected dominating set with $|S| < 3$. Without loss of generality assume that $|S| = 2$. Consider $D = \{X_1, X_2\}$,. Define Ω_i= { α : $I_\alpha \subset F_\alpha$ in $X_i \in$ S}, $i = 1,2$. If $X_1, X_2 \in V_1$, then there exists a vertex of the form I, j ∈ which does not form an edge with any vertex in S. Suppose $X_1, X_2 \in V_2$ or $X_1 \in V_1, X_2 \in V_2$. Proof follows from case 1 and case 2 of Theorem 3.7, which completes the proof.

Reverse inclusion follows from Theorem 3.9 and Theorem 3.7.

In the following Theorem, we show that if R is a product of local ring with $m_i \neq \emptyset$, then $\gamma(\mathcal{H}_3(R)) = n - 2$.

Theorem 3.11. $R = R_1 \times \cdots \times R_n$, n $\geq$ 5 be a ring where each (R_i, m_i) is a local ring, $m_i \neq \emptyset$. Then $\gamma(\mathcal{H}_3(R)) = n - 2$. Further, γ $(\mathcal{H}_3(R))$ = $\gamma_c(\mathcal{H}_3(R))$ = $\gamma^{-1}(\mathcal{H}_3(R))$.

Proof. Let for each $i = 1,2,\ldots,n-2$, $X_i \in V_i$ and $D = \{X_1, \ldots, X_{n-2}\}$, In order to complete the proof, it suffices to show that D is a dominating set for $\mathcal{H}_3(R)$. Let $X_l = \prod_l I_i \times \prod_{n-l} R_j \in D$. Define Ω_1= { α : $I_\alpha \subset F_\alpha$ in X_l}. Then X_l forms an edge with vertices of the form $I_i \times \prod_{n-1} F_j \in V_1$ where $i \notin \Omega_1$, $\prod_2 I_i \times \prod_{n-2} F_j$, where $i \notin \Omega_l$, $I_i \times I_l \times \prod_{n-2} F_j \in V_2$ where $i \in \Omega_l$, $\prod_{2_{i \in \Omega_l}{}^c} I_i \times I_s \times \prod_{n-3} F_j$, $s \in \Omega_l$, $I_i \times \prod_{2_{i\in\Omega_l}} I_i \times \prod_{n-3} F_j \in V_3$, $\prod_{2_{i\in\Omega_l}} I_i \times \prod_{2_{j\notin\Omega_l}} I_j \times \prod_{n-4} F_s \in V_4, \ldots, \prod_{2_{i\in\Omega_l}} I_i \times \prod_{2_{j\notin\Omega_l}} I_j \times \prod_{n-4} F_s \in V_4, \ldots, \prod_{n-4_{i\in\Omega_l}} I_i \times \prod_{2_{i\in\Omega_l}{}^c} I_l \times \prod_2 F_j \in V_{n-2}$. Clearly D is a dominating set and hence γ $(\mathcal{H}_3(R)) \leq$ n − 2.

Suppose $D = \{X_1, \ldots, X_m\}$, is a dominating set with m < n − 2. Suppose $X_i \in V_1$ for all i. Define Ω= { α : $I_\alpha \subset F_\alpha$ in $X_i \in$ S}. Then there exists a vertex Fj which is not dominated by any vertex of S. Suppose $X_i \in V_j$ for all i, 2 ≤ j ≤ n − 2 or S ∩ $V_i \neq \emptyset$ for at least two j . Proof follows from case 1

and case 2 of Theorem 3.7, which completes the proof.

Theorem 3.12. Let $R = R_1 \times \cdots \times R_n$ be a commutative ring with identity where each (R_i, m_i) is a local ring, $m_i \neq \emptyset$. Then γ $(\mathcal{H}_3(R))$ = n − 1 = $\gamma_c(\mathcal{H}_3(R))$ =γ^{-1} $(\mathcal{H}_3(R))$ if and only if $n = 3,4$.

Proof. Assume that $n = 3$ or 4. Consider $D = \{X_1, \ldots, X_{n-1}\}$, where $X_i = I_s \times \prod_{n-1} R_t \in V_1$. We claim that D is a γ - set of $\mathcal{H}_3(R)$ with $|D| = n - 1$. Define Ω_i= { m : $I_m \subset F_m$ in $X_i \in$ D}. Then X_i forms an edge with vertices of the form $I_\alpha \times \prod_{n-1} R_\beta \in V_1, \prod_2 I_\alpha \times \prod_{n-2} R_\beta \in V_2, \ldots, \prod_{n-2} I_\alpha \times \prod_2 R_\beta \in V_{n-2}$, where $\alpha \notin \Omega_i$. Hence γ $(\mathcal{H}_3(R)) \leq$ n − 1.

Suppose $S = \{X_1, \ldots, X_m\}$ is a dominating set with $|S| < n - 1$. Without loss of generality assume that $|S| = n - 2$. Let $X_i \in V_1$. Define $\Omega = \cup\, \Omega_i$ Then X_i cannot cover vertices of the form $\prod_{i\in\Omega_{n-2}} I_i \times \prod_2 R_j$. Thus $|S| \geq n - 1$, which completes the proof.

Reverse inclusion follows from Theorem 3.12.

4. Independent Domination number of $\mathcal{H}_3(R)$.

This section is devoted to the study of independent domination in 3-nil hypergraph of ideals.

Consider next the class of all independent dominating sets in a hypergraph. For example, let $R = F_1 \times F_2 \times F_3 \times F_4$ be a ring, where each F_i is a field. Then $X_1 = F_1 \times F_2 \times F_3 \times (0)$, $X_2 = F_1 \times F_2 \times (0) \times F_4$, $X_3 = F_1 \times (0) \times F_3 \times F_4$, $X_4 = (0) \times F_2 \times F_3 \times F_4$, $X_5 = F_1 \times F_2 \times (0) \times (0)$, $X_6 = F_1 \times (0) \times F_3 \times (0)$, $X_7 = F_1 \times (0) \times (0) \times F_4$, $X_8 = (0) \times F_2 \times F_3 \times (0)$, $X_9 = (0) \times F_2 \times (0) \times F_4$, $X_{10} = (0) \times (0) \times F_3 \times F_4$. Here $D = \{X_1, X_5\}$ and hence $\gamma_i(\mathcal{H}_3(R)) = 2$.

In the following, we determine the independent domination number of the 3-nil hypergraph of ideals of a ring.

Theorem 4.1. Let $R = R_1 \times \cdots \times R_n, n \geq 3$ be a commutative ring where each (R_i, m_i) is a local ring and $\alpha_i's$ be the number of proper ideals of R_i such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Then $\gamma_i(\mathcal{H}_3(R)) = \sum_{i=3}^{n} (\alpha_i \ \alpha_{i+1} \cdots \alpha_n)$.

Proof. Let $D = \{X_1, \cdots, X_m\}$ where $m = \sum_{i=3}^{n} (\alpha_i \ \alpha_{i+1} \cdots \alpha_n)$ and each $X_i \in V_j$ such that $X_i = \prod_{l=1}^{j} I_l \times \prod_{n-j} R_s$, $1 \leq j \leq n-2$. For $m > j$, X_i covers vertices of the form $\prod_{n-j-1_{m>j}} I_m \times \prod_{j+1} R_l$, $\prod_{n-j-1_{m>j}} I_m \times \prod_{l<j} I_s \times \prod_{n-j-l-1} R_t$, $s \in \{1, \cdots, j-1\}$. Clearly X_i cannot form an edge with vertices of the form $\prod_{l=1}^{j} I_l \times \prod_{n-j} R_t$. Hence D must contain this type of vertices in the dominating set. Clearly D is an independent set and it contains $\alpha_{j+1} \ \alpha_{j+2} \cdots \alpha_n$ elements from each V_j, $1 \leq j \leq n-2$. Therefore, D is an independent dominating set.

Suppose $|S| < m$. Without loss of generality assume that $|S| = m - 1$. Since S is independent, every vertex of S is of the form $X_i = \prod_j I_l \times \prod_{n-j} R_t$, $1 \leq j \leq n-2$ such that $\cap X_i = I_l \times \prod_{n-1} R_j$, . Then X_i cannot cover vertex of the form $\prod_j I_s \times \prod_{n-j} R_m$, $1 \leq j \leq n-2$, a contradiction.

The following corollary is nearly immediate in light of the theorem. One can prove the following corollary in analogous to the above.

Corollary 4.2. Let $R = F_1 \times F_2 \times F_3 \times F_4$, $n \geq 4$ be a commutative ring where each F_i is a field. Then $\gamma_i(\mathcal{H}_3(R)) = n - 2$.

To this end, we recall the definition of hyperdomination. Let $\mathcal{H} = (X, E)$ be a hypergraph. A set $D \subseteq X$ is called a hyperdominating set if for each $\in X - D$, there exists an edge $E \in \mathcal{E}$ such that $E - v \subseteq D$. The minimum cardinality of all hyperdominating sets is called the hyperdomination number of $\mathcal{H}$ and is denoted by $\gamma_h(\mathcal{H})$.

For example, if $R = F_1 \times F_2 \times F_3$, then $\gamma_h(\mathcal{H}_3(R)) = 2$. Suppose $R = F_1 \times \cdots \times F_n$, where n = 4 and 5, then $\gamma_h(\mathcal{H}_3(R)) = 4$ and 7 respectively. We conclude this section by posing a question on the hyperdomination number of 3-nil hypergraph of ideals.

Question 4.3. Can we find the hyperdomination number of 3-nil hypergraph of ideals?

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