CERTAIN CLASSES OF UNIVALENT ANALYTIC FUNCTIONS WITH SOME FIXED INITIAL COEFFICIENTS

(1) K.V.Sitavani, (2) V. Srinivas
(1) Research Scholar, (2) Professor
(1) Mathematics Department, Jawaharlal Nehru Technological University, Kakinada, India.
(2) Mathematics Department, Dr. B. R. Ambedkar Open University, Hyderabad, India.

Abstract: In this paper, we find a subclass of univalent analytic functions by fixing second, third, fourth Taylor coefficients. We investigate coefficient bounds, starlikeness, convexity, growth, distortion theorems, and extreme points for this class.

Index Terms: Univalent functions

Introduction

Let S be the class of all functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

which are analytic and univalent in $U = \{z \in C: |z| < 1\}$.

Let T be the subclass [4] of S of all functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n , a_n \geq 0 \quad (1)$$

for $z \in U$.

A function $f(z) \in T$, is said to be starlike[1] of order α if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha , \ 0 \leq \alpha < 1$$

A function $f(z) \in T$, is said to be convex [1] of order α if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \alpha, \ 0 \leq \alpha < 1$$

Silverman [2] proved that if $f(z)$ given by (1) is in T and $a_2 > 0$ then a sufficient condition for $f(z)$ to be in T is given by

$$\sum_{n=3}^{\infty} n(n-1)a_n \leq 2a_2 \quad (2)$$

Now we introduce a subclass [3] $T(b, d, B_n)$ of T by fixing a_2, a_3 *and* a_4 by imposing a generalized form of the condition (2) as follows:

$$T(b, d, B_n) = \{f \in T: f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n\} \ \sum_{n=4}^{\infty} B_n a_{n+1} \leq (2b - dB_3) \quad (3)$$

where $0 \leq b \leq \frac{1}{4}, 0 \leq d \leq \frac{1}{24}, \ B_n \geq n(n+1)$.

Section 1

In section 1, we find a coefficient characterization for $T(b, d, B_n)$, a sufficient condition for starlkeness and a condition for functions in this class to be convex of order α.
First we find a necessary condition for functions in $T(b, d, B_n)$ in terms of Taylor coefficients.

Theorem 1: For $0 \leq b \leq \frac{1}{4}, 0 \leq d \leq \frac{1}{24}$, $z \in U$, a function

$f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n \in T(b, d, B_n)$ if and only if

$$\sum_{n=4}^{\infty} n(n+1)a_{n+1} \leq 2b - 12d .$$

Proof: Let $f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n$ be function defined in the class $T(b, d, B_n)$

we have $\sum_{n=4}^{\infty} B_n\, a_{n+1} \leq 2b - dB_3$

$$\sum_{n=4}^{\infty} n(n+1)\, a_{n+1} \leq 2b - 12d$$

since $B_n \geq n(n+1)$

Hence $a_{n+1} \leq \frac{2b-12d}{n(n+1)}$, $n \geq 4$

or $a_n \leq \frac{2b-12d}{n(n-1)}$, $n \geq 5.$

This completes the proof.

conversely,

$\sum_{n=4}^{\infty} n(n+1)a_{n+1} \leq 2b - 12d$, $0 \leq b \leq \frac{1}{4}, 0 \leq d \leq \frac{1}{24}.$

$\sum_{n=2}^{\infty} na_n = 2b + 4d + \sum_{n=5}^{\infty} a_n z^n \leq 2b + 4d + \sum_{n=5}^{\infty} \frac{2b-12d}{n(n+1)} z^n \leq 1$

$\therefore f \in T$ and hence $f \in T(b, d, B_n)$.

Also, $\sum_{n=4}^{\infty} B_n a_{n+1} \leq n(n+1)\frac{2b-12d}{n(n+1)} = 2b - 12d.$

This completes the proof.

In the following result we find sufficient condition for starlikeness of the class $T(b, d, B_n)$.

Theorem 2: A function $f \in T(b, d, B_n)$, $0 \leq b \leq \frac{1}{4}, 0 \leq d \leq \frac{1}{24}$ is said to be starlike of order α for some $0 \leq \alpha < 1$ if

$$\sum_{n=5}^{\infty} (n-\alpha)|a_n| \leq (1-\alpha) - (2-\alpha)b - (4-\alpha)d$$

Equality occurs for $(z) = z - bz^2 - dz^4 - \frac{(1-\alpha)-(2-\alpha)b-(4-\alpha)d}{n-\alpha} z^n$, $z \in U$.

Proof: Let $f \in T(b, d, B_n)$ is said to be starlike of order α if and only if

$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha$, $0 \leq \alpha < 1$ for $z \in U$. This is obtained by

$$\left|\frac{zf'}{f} - 1\right| \leq 1 - \alpha, z \in U$$

For $z \in U$, and $f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n$ we have

$$f'(z) = 1 - 2bz - 4dz^3 - \sum_{n=5}^{\infty} na_n z^{n-1} \qquad (4)$$

these gives $\left|\frac{zf'}{f} - 1\right|$

$$= \left|\frac{z - 2bz^2 - 4dz^4 - \sum_{n=5}^{\infty} na_n z^n}{z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n} - 1\right|$$

$$= \left|\frac{z - 2bz^2 - 4dz^4 - \sum_{n=5}^{\infty} na_n z^n - z + bz^2 + dz^4 + \sum_{n=5}^{\infty} a_n z^n}{z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n}\right|$$

$$= \left|\frac{-bz^2 - 3dz^4 - \sum_{n=5}^{\infty}(n-1)a_n z^n}{z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n}\right|$$

$$\leq \frac{br + 3dr^3 + \sum_{n=5}^{\infty}(n-1)a_n r^{n-1}}{1 - br - dr^3 - \sum_{n=5}^{\infty} a_n r^{n-1}}$$

$$\leq \frac{b + 3d + \sum_{n=5}^{\infty}(n-1)a_n}{1 - b - d - \sum_{n=5}^{\infty} a_n}$$

Now the right hand side expression in above inequality is at the most $1 - \alpha$ if

$$b + d + \sum_{n=5}^{\infty}(n-1)a_n \leq (1-\alpha)(1 - b - d - \sum_{n=5}^{\infty} a_n)$$

This is the given condition. Hence completes the proof.

In the next result, we find a sufficient condition for convexity for the function in the class$T(b, d, B_n)$.

Theorem 3: A function $f \in T(b, d, B_n)$, $0 \leq b \leq \frac{1}{4}, 0 \leq d \leq \frac{1}{12}$

is convex of order α if

$$\sum_{n=5}^{\infty} n(n-\alpha)|a_n| \leq (1-\alpha) - 2(2-\alpha)b - 4(4-\alpha)d$$

$0 \leq \alpha < 1$ and $z \in U$.

Equality occurs when $f(z) = z - bz^2 - dz^4 - \frac{(1-\alpha)-2(2-\alpha)b-4(4-\alpha)d}{n(n-\alpha)} z^n, z \in U$.

Proof: A function

$f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n \in T(b, d, B_n)$ is said to be convex of order α if and only if

$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \alpha,\ 0 \leq \alpha < 1$

This is given by

$$\left|\frac{zf''}{f'}\right| \leq 1 - \alpha \qquad (5)$$

Equation (4) gives

$f''(z) = -2b - 12dz^2 - \sum_{n=5}^{\infty} n(n-1)a_n z^{n-2}$

left hand side of the inequality (5) is

$$\left|\frac{-2bz-12dz^3-\sum_{n=5}^{\infty} n(n-1)a_n z^{n-1}}{1-2bz-4dz^3-\sum_{n=5}^{\infty} na_n z^{n-1}}\right|$$

$$\leq \frac{2br+12dr^3+\sum_{n=5}^{\infty} n(n-1)a_n r^{n-1}}{1-2br-4dr^3-\sum_{n=5}^{\infty} na_n r^{n-1}}$$

$$\leq \frac{2b+12d+\sum_{n=5}^{\infty} n(n-1)a_n}{1-2b-4d-\sum_{n=5}^{\infty} na_n} \leq 1-\alpha$$

by the given condition

$$\sum_{n=5}^{\infty} n(n-\alpha)|a_n| \leq (1-\alpha)-2(2-\alpha)b-4(4-\alpha)d$$

.

This completes the proof.

Section 2

In this section we discuss growth, distortion and extreme points for the functions of the class $T(b, d, B_n)$.

Now we find growth bounds for the functions in the class $T(b, d, B_n)$ with increasing B_n.

Theorem 4: For $B_4 \geq \frac{5(2b-12d)}{1-2b-4d}$ and $0 \leq b \leq \frac{1}{2}, 0 \leq d \leq \frac{1}{4}$

Let $\{B_n\}$ be a non-decreasing sequence with $B_n \geq n(n+1)$ for $n \geq 4$. Then for $f \in T(b, d, B_n)$ we have

$$\max\left\{0, r-br^2-dr^4-\frac{2b-12d}{B_4}r^5\right\} \leq |f(z)| \leq r+br^2+dr^4+\frac{2b-12d}{B_4}r^5.$$

where $|z| = r, z \in U$.

Equality occurs when $f(z) = z-bz^2-dz^4-\frac{2b-12d}{B_4}z^5$.

Proof: For $z \in U$, $f(z) = z-bz^2-dz^4-\sum_{n=5}^{\infty} a_n z^n \in T(b, d, B_n)$

$B_4 \sum_{n=5}^{\infty} a_{n+1} \leq \sum_{n=4}^{\infty} B_n a_{n+1} \leq 2b-12d,$

since $\sum_{n=5}^{\infty} a_{n+1} \leq \frac{2b-12d}{B_4}$

Further,

$$|f(z)| \geq \max\{0, |z|-b|z|^2-d|z|^4-|z|^5 \sum_{n=5}^{\infty} a_n \qquad (6)$$

$$\geq \max\{0, r-br^2-dr^4-r^5 \frac{2b-12d}{B_4}\}$$

and

$$f(z) \leq r+br^2+dr^4+r^5 \sum_{n=5}^{\infty} a_n \leq r+br^2+dr^4+\frac{2b-12d}{B_4}r^5$$

This and (6) together complete the proof.

Now we find distortion bounds for the functions in the class $T(b,d,B_n)$ with increasing B_n.

Theorem 5: For $B_4 \geq \frac{5(2b-12d)}{1-2b-4d}$ and $0 \leq b \leq \frac{1}{2}, 0 \leq d \leq \frac{1}{4}$

$f \in T(b,d,(n+1)B_n)$ and $B_n \leq B_{n+1}$, then

$$\max\left\{0, 1-2br-4dr^3-\frac{2b-12d}{B_4}r^4\right\} \leq f'(z) \leq 1+2br+4dr^3+\frac{2b-12d}{B_4}r^4$$

where $|z| = r, z \in U$.

Equality occurs when $f_4(z) = z - bz^2 - dz^4 - \frac{2b-12d}{5B_4}z^5$

Proof: For $z \in U$, $f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n \in T(b,d,B_n)$

We have $B_4 \sum_{n=5}^{\infty} na_n \leq \sum_{n=4}^{\infty} B_n a_{n+1} \leq 2b - 12d$

From (4)

$$|f'(z)| \geq \max\{0,1-2br-4dr^3-\sum_{n=5}^{\infty} na_n\} \geq \max\{0,1-2br-4dr^3-\frac{2b-12d}{B_4}r^4\}$$

and

$$|f'(z)| \leq 1+2br+4dr^3+r^4\sum_{n=5}^{\infty} na_n \leq 1+2br+4dr^3+r^4\frac{2b-12d}{B_4}$$

Where $|z| = r$. This completes the proof.

In the next theorem, we discuss the extreme points of the class $T(b,d,B_n)$.

Theorem 6: The class $T(b,d,B_n)$ is a convex subfamily of T.

Proof: For $f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n$ and $g(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} b_n z^n$

belonging to $T(b,d,B_n)$ we have

$h(z) = \lambda f(z) + (1-\lambda)g(z)$

$=\lambda[z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n] + (1-\lambda)[z - bz^2 - dz^4 - \sum_{n=5}^{\infty} b_n z^n]$

$= z - bz^2 - dz^4 - \sum_{n=5}^{\infty}[\lambda a_n + (1-\lambda)b_n]z^n$

$= z - bz^2 - dz^4 - \sum_{n=5}^{\infty} A_n z^n$.

Since f and g are in $T(b,d,B_n) \subseteq T$ and T is convex [3] hence $h \in T$.

$$\sum_{n=4}^{\infty} B_n A_{n+1} = \lambda\sum_{n=4}^{\infty} B_n a_{n+1} + (1-\lambda)\sum_{n=4}^{\infty} B_n b_{n+1} \leq 2b - 12d$$

Because f and g are in $T(b,d,B_n)$. Thus $h \in T(b,d,B_n)$ and the theorem is proved.

Following result characterizes function in $T(b, d, B_n)$.

Also, we discuss the extreme of points of the class $T(b, d, B_n)$.

Theorem 7: Let $B_k \geq \frac{(k+1)(2b-12d)}{(1-2b+12d)} > 0$. $0 \leq b \leq \frac{1}{4}, 0 \leq d \leq \frac{1}{12}$ $f_3(z) = z - bz^2 - dz^4$ and

$f_n(z) = z - bz^2 - dz^4 - \frac{2b-12d}{B_n} z^{n+1}, n \geq 4$ are the extreme points. Then $f \in T(b, d, B_n)$ if and only if f(z) can be expressed as

$f(z) = \sum_{n=3}^{\infty} \lambda_n f_n(z)$

Where $\lambda_n \geq 0$ for $n \geq 4$ and $\sum_{n=3}^{\infty} \lambda_n = 1$.

Proof: Suppose that f(z) can be expressed as (4). Then we have

$$f(z) = \sum_{n=3}^{\infty} \lambda_n f_n(z) = z - bz^2 - dz^4 - \sum_{n=4}^{\infty} \frac{(2b-12b)\lambda_n}{B_n} z^{n+1}$$

which can be expressed as $f(z) = z - \sum_{n=2}^{\infty} A_n z^n$ where $A_2 = b, A_3 = 0, A_4 = d, A_n = \frac{(2b-12b)\lambda_n}{B_n}$ for $n \geq 5$.

The function f(z) is analytic in U. Since, $\sum_{n=4}^{\infty} nA_n \leq 1$, we have $f \in T$.

Further $\sum_{n=4}^{\infty} B_n A_{n+1} \leq 2b - 12d$,

Hence we have $f \in T(b, d, B_n)$.

Conversely, suppose that $f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n$ belongs to the class $T(b, d, B_n)$

Since $\sum_{n=4}^{\infty} B_n a_{n+1} \leq 2b - 12d$, for $n \geq 4$ we may put

$\lambda_n = \frac{B_n a_{n+1}}{2b-12d}, (n \geq 4)$ and

$\lambda_3 = 1 - \sum_{n=4}^{\infty} \lambda_n$. Therefore, we have

$f(z) = z - bz^2 - dz^4 - \sum_{n=5}^{\infty} a_n z^n$

$= \lambda_3(z - bz^2 - dz^4) + \sum_{n=4}^{\infty} \lambda_n \left(z - bz^2 - dz^4 - \frac{2b-12d}{B_n} z^{n+1}\right)$

$= \sum_{n=3}^{\infty} \lambda_n f_n(z)$.

This completes the assertion of the theorem.

References:

1 .A.W.Goodman: A Text book on, "Univalent Functions"

2. H.Silverman : Univalent Functions having univalent Derivatives, Rocky Mountain J. Maths, 16(1986), pp. 55-61.

3. V. Srinivas: On Certain Classes of Analytic Functions with Fixed Second Coefficient,

J. Ramanujan Math.Soc 6 Nos 1& 2 1991, PP 151-166.

4. H. Siverman: Univalent Functions with negative coefficients.Vol.51,No: 1, (1975),pp.109-116.