Iterated Function Systems as a Generator of Fractal Objects.

[1]Arun Mahanta, [2]Hemanta Kr. Sarmah and [3]Gautam Choudhury,
[1] Department of Mathematics, Kaliabor College, Nagaon, Assam, India
[2] Department of Mathematics, Gauhati University, Guwahati, Assam, India
[3] Mathematical Science Division, Institute of Advance Study in Science and Technology, Boragaon, Guwahati, Assam, India

Abstract: In this paper we have done some investigation on some concepts of the theory of metric space to analyze fractal objects. If we iteratively apply a finite set of contraction mappings to any point on a compact metric space, we will come arbitrarily close to a set of points in the space which is very often fractal. The present work addresses the problem of how iterated function systems may be used to construct such fractal objects. For this purpose, we discuss two algorithms producing fractals, namely that of deterministic algorithm, and random iterated algorithm. We have also discussed about the connection between Hausdorff dimension and iterated function systems.

Key words: Affine transformation, Attractor, Contraction factor, Contractive mapping, Fractal, Haudorff dimension, Iterated Function Systems, Metric space.

Introduction:

We usually beleve that the shape of natural objects are determined by its characteristics scale and no new features of it revealed if we try to magnify it beyond that characteristic scale. To measure length, area, volume or other properties of the object we try to measure it at such a resolution which is finer than its characteristics resolution and we expect the value obtained by such method is a unique one for the object. The concept of Euclidean geometry is based on this simple idea.
However, B.B. Mandelbrot brought to our attention that many objects in nature simply do not have this pre-conceived form [14],[15]. In fact, many of the structure in nature such as coast line of a courtry , surface of a mountain or clouds have a very different form. As we zoom into these structures, new and ever finer structures are continuously revealed. When we measure a property like length, area, or volume the computed value depends on how many such finer features of the structure are included in our computation. As a result, the values we measure actually depends on the spatial ruler we used to compute our measurement. The study of such objects has resulted in a new area of mathematics called *Fractal Geometry*. Fractal geometry was popularized by the mathematician Benoit Mandelbrot, and it was he who coined the term fractal in 1977. Though the mathematical work of fractal geometry was first initiated by Cayley, Fatou and Julia in the late 19th and early 20th centuries[9], progress of research in this line was slow until the development of the electronic computer. Much of the current interest in fractals is a consequence of Mandelbrot's work. His computer simulations of maps of the complex plane have resulted in extremely complicated and beautiful fractals [16].

A fractal is a complex geometric shape with details down to the smallest spatial scales. Fractals are self similar, whereby a subset of a fractal, and any subset thereof, may resemble the fractal as a whole e.g. see Falconer [8]. Consequently, a measure of the area of a fractal is often difficult to determine. For illustration let us begin with a simple example, the Sierpinski Carpet. The process of generation of the Sierpinski Carpet is that one starts with the unit box and divide it into nine equal boxes, and then remove the open central box. This process is repeated for each eight remaining sub boxes. The limiting set is a fractal which is a generalization of the two dimensional cantor set. Although it can be shown that the Sierpinski Carpet has zero area, it is still useful to make some kind of determination of its dimension.

The values of the measured properties of many physical systems look random. It is believed that such random looking fluctuations must be the result of mechanisms driven by chance. However, today it was an established fact that everything that looks random may not be random in reality. On the other hand, there are dynamical systems where the fluctuations of the values of the variables are so complex that they mimic random behavior, for example see [10] . The reason for this type of behavior is now called 'chaos', which may occur even in the simplest of physical systems. This is one of the major achievements of mathematics in last few years, the recognition that simple, deterministic physical or mathematical systems may behave unpredictably or randomly. The great enthusiasm of bonding the concept of chaos and fractals is to change our mind set. These concepts act as an agent to nag us for thinking of alternative approaches. The mathematical ideas behind fractal and chaos augment the set of analytical tools we have from our childhood. When we find the experimental values of the measured properties of some physical system look random, we usually conclude that this must be the result of mechanism driven by chance. However, now a days there are methods to analyze random like experimental data to determine whether the data could have been generated by some deterministic process[19]. If this is true , further, the analysis is able to reconstruct the mathematical form of the deterministic relationship. The mathematics of 'non linear dynamical system' is based on this methods and they use many of the idea and properties of fractals.

Geometry is concerned with making our understanding about shape, size, position, area etc. of physical world, and fractal geometry extend that process. The theory of Iterated Function Systems (IFS) provides us new method to describe many of the natural fractals like leaf of a tree, fern, clouds etc. as clearly as an architect can describe a house. It express relation between parts of a generalized fractal objects by using classical geometical entities like affine transformation, scaling, shrinking, twiesting, shifting, contraction mapping etc. Using only these relations IFS theory defines and conveys intricate fractal structures of some natural objects.

Barnsley and Demko first introduce the term 'Iterated fuction systems' to represent a method for generating fractals [2] but the essential concept is usually attributed to Hutchinson[12].The feasibility of using IFS in computer graphics was reviewed first at the SIGGRAPH meeting by Demko, Naylor, and Hodges [6]. The use of fractal geometry, both deterministic and non deterministic, to model natural objects, has been investigated by a number of authors, including Mandelbrot[6,7], Miller[17], Kawaguchi[13], Smith[21], Openheimer[18], Ambum et. al.[1], Fournier et. al.[11]. A detailed introduction to IFS's is prsented in [3]. The approach presented in this paper has its roots in these investigations.

The rest of the paper is organized as follows: in section 2, we provide a review of some definitions and preliminary concepts concerning metric spaces . In section 3, we have described about a mathematical space where we intended to work with together with some concepts of contraction mappings which is an important tool for our further investigation. Section 4 contains discussion about the effect of a finite set of contraction mappings acting iteratively on a point of the space defined in section 3 and hence find two algorithms for creating fractal objects. Section 5 concern with a method of finding Hausdorff dimension of the fractals produced by the method discussed in section 4 Finally, in section 6, we have given a concluding remark of our study.

2. Preliminaries:

Topology, a branch of mathematics first formalized by France mathematician Henri Poincare by his 1895 publication 'Analysis Situs'. The main topic of interest here are the properties that remain unchanged by continuous deformation like bending, twisting, stretching or shrinking. A metric space is a set for which distances between all members of the set defined. Those distances, taken together, called a metric on the set. A metric on a space induces topological properties like open and closed set, which leads to the study of more abstract topological spaces. In order to carry our study, we first need to provide some definitions concerning metric space, which are discussed in this section.

Definition 2.1[20]: A *metric space* (X,d) consists of two objects, a non-empty set X together with a real function d of ordered pairs of elements of X which satisfies the following three conditions for all $x,y,z \in X$:

i. $d(x,y) \geq 0$, and $d(x,y) = 0 \Leftrightarrow x = y$;

ii. $d(x,y) = d(y,x)$;

iii. $d(x,y) \leq d(x,z) + d(z,y)$.

Definition 2.2[20] : A sub set G of the metric space (X,d) is called an *open set* if, given any point y in G, there exist a positive real number r such that $S_r(y) \subseteq G$ where, $S_r(y) = \{x \in X : d(x,y) < r\}$ is the open sphere center at y and radius r. That is, G is open if each point of it is the center of some open sphere contained in G .

Definition 2.3[20] : A point $x \in X$, where (X,d) is a metric space, is called a *limit point* of the sub set A of X if each open sphere centered on x contains at least one point of A different from x.

Definition 2.4[20] : A subset F of the metric space (X,d) is called *closed set* if it contains each of its limit points.

Definition 2.5[7] : A subset F of the metric space (X,d) is called *bounded* if there is an element x in X and a real number $M > 0$ such that for each element a in F, $d(a,x) \leq M$.

Definition 2.6[20] : A sequence $\langle a_n \rangle_{n=1}^{\infty}$ of points in a metric space (X,d) is called a *Cauchy sequence* if, for any given number $\varepsilon > 0$, there is an integer $N_0 > 0$ such that $d(a_n, a_m) < \varepsilon$ for all $n, m \geq N_0$.

Definition 2.7[20] : A sequence $\langle a_n \rangle_{n=1}^{\infty}$ of points in a metric space (X,d) is said to *converge to a point* $a_0 \in X$ if, for any given number $\varepsilon > 0$, there is an integer $N_0 > 0$ such that $d(a_n, a_0) < \varepsilon$ for all $n \geq N_0$. The point a_0 is called limit of the sequence $\langle a_n \rangle_{n=1}^{\infty}$.

Definition 2.8[20]: A metric space (X,d) is *complete* if every Cauchy sequence $\langle a_n \rangle_{n=1}^{\infty}$ in X converges to a point in X .

Definition 2.9[20]: A subset A of a metric space (X,d) is *compact* if every infinite sequence $\langle a_n \rangle_{n=1}^{\infty}$ of points in A contains a subsequence having a limit in A.

Theorem 2.1 (Generalized Heine-Borel Theorem): Every closed and bounded subspace of R^n , where R is the set of real numbers, is compact.

Proof: The proof of this theorem can be found in [20].

Definition 2.10[20] : Let (X,d) be a metric space, then the distance from the point $x \in X$ to the compact sub set A of X is defined as

$$d_1(x,A) = \min\{d(x,a) : a \in A\}$$

Definition 2.11[20] : The *distance between two compact subsets* A *and* B of the metric space (X,d) is defined as

$$d_2(A,B) = \max\{d_1(x,B) : x \in A\}$$

This notion of distance is not symmetric as shown in the *figure 2.1.*

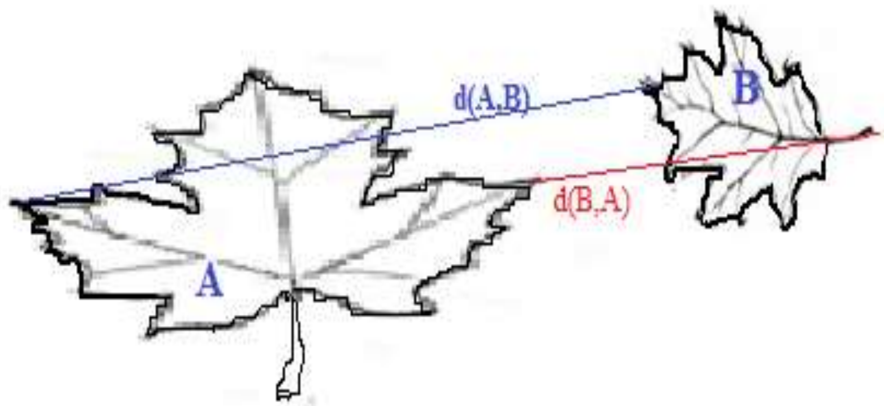


Figure 2.1 [*Distance between two compact sets*]

Definition 2.12 : A *linear transformation* from $\mathbb{R}^2$ to itself is a function f from $\mathbb{R}^2$ to $\mathbb{R}^2$ with the properties: $f(x+y) = f(x) + f(y)$ and $f(a\,x) = a\,f(x)$, where x, y are vectors in $\mathbb{R}^2$ and a is any real number. In other words, the map f is linear if it can be put in the form

$f(x, y) = (ax + by,\ cx + dy)$ i.e., in matrix notation f can be expressed as:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$, where, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix in $\mathbb{R}$.

Definition 2.13: A transformation f from $\mathbb{R}^2$ to $\mathbb{R}^2$ of the form

$$f(x, y) = (ax + by + e,\ cx + dy + f)$$

where a, b, c, d, e, f are reals is called a two dimensional *affine transformation*.

In matrix notation it can be put in the form, $f(X) = AX + t$ where, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $t = \begin{pmatrix} e \\ f \end{pmatrix}$.

Note that an affine transformation consists of a linear transformation followed by a translation.

Definition 2.14: An affine transformation that preserves angles is called a *similitude*. It is formed by any combination of the following actions:

(a) **Translation:** A translation T moves points by a fixed vector (e, f) i.e.,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

(b) **Rotation:** A rotation T rotates the points around a fixed point through an angle θ i.e.,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(c) **Reflection:** If x and y be two points lying opposite sides of the line $y = mx$ such that the line $y = mx$ is the perpendicular bisector of the line segment joining x and y, then a reflection through the line $y = mx$ is the linear transformation T which exchanges these two points and leave points on the line $y = mx$ unmoved, i.e.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(d) **Dilation:** The transformation T action of which multiply the coordinates of all points on which it acts by a constant, say $'c'$ is called dilation i.e.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Note that if T is a similitude then A and $T(A)$ have the same shape but not necessarily the same size.

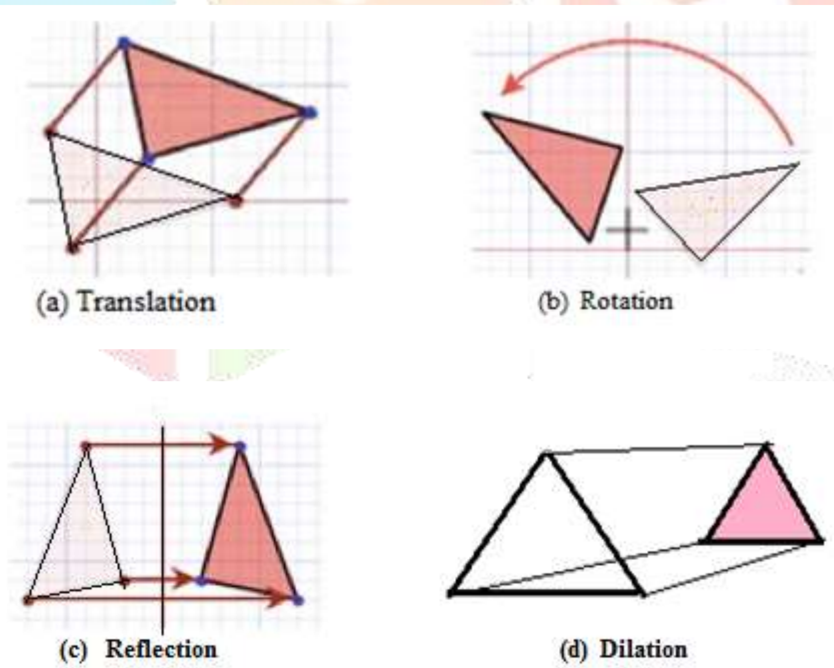


Figure 2.2 [*Action of some basic similitudes*]

Here we note that in a similitude T all distances are treated as $d(T(x), T(y)) = c\,d(x, y)$.

3. The Space of Fractal Objects:

As our prime interest is to study images of fractal object we first try to define a space whose elements are images. Now every image is nothing but collection of some points in Euclidean space $\mathbb{R}^2$ which are compact. Thus we consider the set of all possible compact subsets of $\mathbb{R}^2$, in general all possible compact subsets of a metric space. Therefore, for the metric space (X, d) we consider the set

$$\mathbf{H}(X) = \{A \subset X,\ A \neq \phi, A \text{ is compact}\}$$

Now we need to define a metric on $\mathbf{H}(X)$. Since the distance between two compact sets A and B as defined in definition 2.11 is asymmetrical, it does not serve our purpose. Therefore, we choose the Hausdorff metric for this purpose which is defined below:

Definition 3.1: For any $A, B \in \mathbf{H}(X)$, we define the *Hausdorff metric* $d_{\mathbf{H}} : X \times X \to R$ by

$$d_{\mathbf{H}}(A,B) = \max\{d_2(A,B), d_2(B,A)\}$$

An easy way to understand the meaning of Hausdorff metric is through the concept of $\delta - collar$ of a set which is given in figure 3.1, where $d_{\mathbf{H}}(A,B) = d$.

Definition 3.2: Let A be a subset of a metric space (X,d) and $\delta > 0$, then the $\delta - collar$ of the set A is the set A_δ defined by

$$A_\delta = \{x : d(x,a) < \delta \text{ for some } a \in A\}.$$

Note that, $A \subseteq A_\delta$ for any $\delta > 0$. For any $A, B \in \mathbf{H}(X)$ if $A \subseteq B_\delta$ then there exist $b \in B$ such that $d(a,b) < \delta$ for all $a \in A$. Which confirms that $\inf\{d(a,b) : b \in B\} < \delta$, that is $d_1(a, B) < \delta$. Taking supremum of all such $d_1(a,B)$ for $a \in A$ we have

$$Sup\{d_1(a, B) : a \in A\} \le \delta$$, which shows that $d_2(A, B) \le \delta$.

Again for $B \subseteq A_\delta$, with similar argument one can show that $d_2(B, A) \le \delta$. Thus the minimum of such $\delta > 0$ must represent the Haudorff metric $d_{\mathbf{H}}$.

Definition 3.3: The Hausdorff metric for any $A, B \in \mathbf{H}(X)$ is

$$d_{\mathbf{H}}(A,B) = \inf\{\delta : A \subseteq B_\delta \text{ and } B \subseteq A_\delta\}.$$

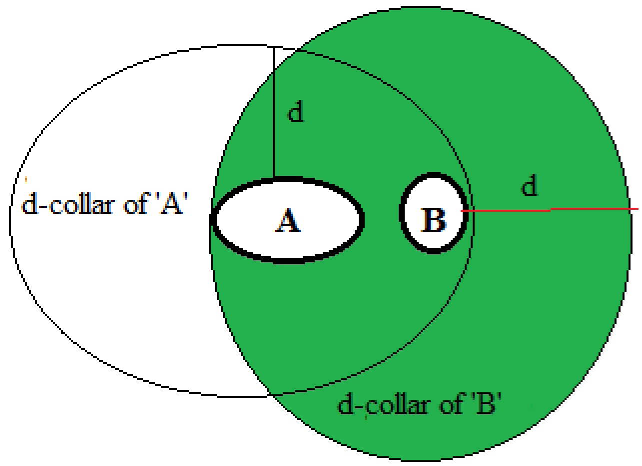


Figure 3.1 [*Graphical exibition of Hausdorff metric*]

Theorem 3.1: Let (X,d) be a metric space. Then $(\mathbf{H}(X), d_{\mathbf{H}})$ is also a metric space.

Proof: We have for each $a \in A$ the minimum distance of a from an arbitrary point x of A is zero and thus $d_1(x,A) = 0 \ \ \forall\ x \in A$ which imply that for all $A \in \mathbf{H}(X)$, $\max\{d_1(x,A):\ x \in A\} = 0$, *i.e.* $d_H(A,A) = 0$. If $A, B \in \mathbf{H}(X)$ such that $A \neq B$ then either $A \subset B$ or $B \subset A$ .. Now,

$A \subset B \Rightarrow d_2(B,A) > 0$ and $B \subset A \Rightarrow d_2(A,B) > 0$ and hence $d_{\mathbf{H}}(A,B) > 0$.

Thus $d_{\mathbf{H}}(A,B) = d_{\mathbf{H}}(B,A)$ is a non-negative real number.

To prove triangular inequality for $d_{\mathbf{H}}$ we first prove it for d_2 .

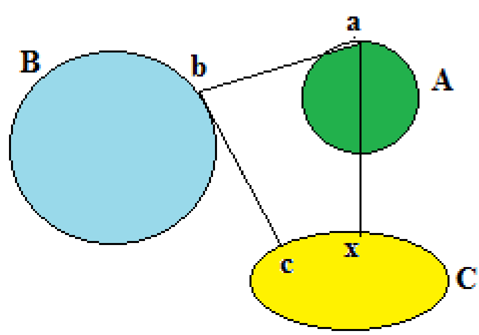


Figure 3.2

Let 'a' be the point on the set A whose distance from any point on the set C is maximum and $x \in C$ be closest to 'a' , as shown in the figure 3.2.

$$\therefore\ d_2(A,C) = d(a,x) \qquad \cdots \qquad (3.1)$$

If $b \in B$ be the closest point from 'a' and $c \in C$ be the closest point to 'b' then

$$\therefore\ d_1(a,B) = d(a,b) \qquad \cdots \qquad (3.2)$$

$$and\ d_1\ (b\ ,\ C)\ =\ d\ (b,\ c) \qquad \cdots \qquad (3.3)$$

Therefore, $\therefore\ d_2\ (A,B) \geq d_1(a,B) = d\ (a,b) \qquad \cdots \qquad (3.4)$

$$and\ d_2\ (B,C) \geq d_1(b,C) = d\ (b,c) \qquad \cdots \qquad (3.5)$$

Now,

$$\therefore\ d_2\ (A,B) + d_2(B,C) \geq\ d\ (a,b) + d(b,c) \geq d(a,c) \geq\ d(a,x)\ =\ d_2(A\ ,\ C)$$

Hence,

$$d_{\mathbf{H}}(A,B) + d_{\mathbf{H}}(B,C) \geq d_2(A,B) + d_2(B,C) \geq\ d_2(A,C)$$

Also, $d_{\mathbf{H}}(A,B) + d_{\mathbf{H}}(B,C) = d_{\mathbf{H}}(C,B) + d_{\mathbf{H}}(B,A) \geq d_2(C,B) + d_2(B,A) \geq d_2(C,A)$

Therefore, $d_{\mathbf{H}}(A,B) + d_{\mathbf{H}}(B,C)\ \geq\ \max\{\ d_2(A,C)\ ,\ d_2(C,A)\ \} = \ d_{\mathbf{H}}(A,C)$. ■

Theorem 3.2: If the metric space (X,d) is complete then $H(X)$ is also a complete with respect to the Hausdorff metric $d_{\mathbf{H}}$.

The proof of this theorem can be found in [3].

Till now we have established a metric space, viz. $H(X)$, where we intended to work with. For better understanding and to make use of $H(X)$ we need another tool in the theory of metric space which are discussed below.

Definition 3.4: A mapping $T : X \to X$, where $(X\ ,\ d)$ is a metric space is called a contraction mapping if

$$d(T(x),T(y)) \leq\ c\,d(x,y)$$

For some positive constant $c < 1$ and for all $x, y \in X$. Here c is called contraction factor of the map T .

Theorem 3.3: Contraction mapping on a metric space is continuous.

Proof: Let $(X\ ,\ d)$ be a metric space and $T : X \to X$ be a contraction mapping with contraction factor ' c '.

Let $\varepsilon > 0$ be given. We choose $\delta = \dfrac{\varepsilon}{c}$, then for all $x\ ,\ y \in X$, with $d(\,x, y\,) < \delta$, we have

$$d(T(x),T(y)) \leq\ c\,d(x,y) < c\,\delta = \varepsilon$$

Hence, T is continuous on X . ■

Theorem 3.4 (Contraction mapping): Contraction mapping on a complete metric space has a unique fixed point.

Proof: Let $T : X \to X$ be a contraction mapping on the complete metric space $(X\ ,\ d)$. Therefore, there exist a positive real number $\lambda < 1$ such that

$$d(T(x),T(y)) \leq\ \lambda\,d(x,y)\quad \forall\ x,y \in\ X\ .$$

For any $x_0\ \in X$, we define the sequence $\langle x_n \rangle_{n\,=\,0}^{\infty}$ in X by

$$x_n\ =\ T(x_{n\,-\,1})\ \ ,\ \ n \geq 1$$

First we show that $\langle x_n \rangle_{n\,=\,0}^{\infty}$ is a Cauchy sequence.

Let $m\ ,\ n$ be two positive integers with $m < n$, then we have

$$\begin{aligned} d\left(x_m\ ,\ x_n\right) &\leq\ d\ (x_m\ ,x_{m+1}) + d(x_{m+1}\ ,x_{m+2}) + \cdots + d(x_{n\,-\,1}\ ,x_n) \\ &\leq d\left(T^m(x_0),T^{m+1}(x_0)\right) + d\left(T^{m+1}(x_0),T^{m+2}(x_0)\right) +\ \cdots\ + d\left(T^{n-1}(x_0),T^n(x_0)\right) \end{aligned}$$

$$\leq \lambda^m d\left(x_0, T(x_0)\right) + \lambda^{m+1} d\left(x_0, T(x_0)\right) + \cdots + \lambda^{n-1} d\left(x_0, T(x_0)\right)$$

$$= \lambda^m \left[\sum_{i=0}^{n-m-1} \lambda^i \right] d\left(x_0, T(x_0)\right)$$

$$\leq \lambda^m \left[\sum_{i=0}^{\infty} \lambda^i \right] d\left(x_0, x_1\right)$$

$$= \frac{\lambda^m}{1-\lambda} d\left(x_0, x_1\right)$$

Which shows that $\langle x_n \rangle_{n=0}^{\infty}$ is a Cauchy sequence in X. As the metric space (X, d) is complete, there exist a point $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, $T(x^*) = T\left(\lim_{n\to\infty} x_n\right)$. By continuity of T, we have

$$T(x^*) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$$

i.e., x^* is a fixed point of T.

For uniqueness, we assume that x^* and x^{**} be two fixed points of T.

$$d\left(x^*, x^{**}\right) = d\left(T(x^*), T(x^{**})\right) \leq \lambda\, d\left(x^*, x^{**}\right)$$

As $0 < \lambda < 1$, this is possible only when $d\left(x^*, x^{**}\right) = 0$ which implies that $x^* = x^{**}$.

This completes the proof. ■

Theorem 3.5: If T is a contraction map on a metric space X, then $T: H(X) \to H(X)$ is also a contraction map as set-wise function.

Proof: Since $T: X \to X$ is a contraction map, there exist a real number $c < 1$ such that

$$d(T(x), T(y)) \leq c\, d(x, y)$$

Now for any two compact subset A and B of X and $a \in A$,

$$\begin{aligned} d_1(T(a), T[B]) &= \min\{ d(T(a), y) : y \in T[B] \} \\ &= \min\{ d(T(a), T(b)) : b \in B \} \quad \text{where, } y = T(b) \\ &\leq \min\{ c\, d(a, b) : b \in B \} \\ &= c \min\{ d(a, b) : b \in B \} \\ &= c\, d_1(a, B) \end{aligned}$$

Now for the metric d_2,

$$\begin{aligned} d_2(T[A], T[B]) &= \max\{ d_1(x, T[B]) : x \in T[A] \} \\ &= \max\{ d_1(T(a), T[B]) : a \in A \} \\ &\leq \max\{ c\, d_1(a, B) : a \in A \} \\ &= c \max\{ d_1(a, B) : a \in A \} \\ &= c\, d_2(A, B) \end{aligned}$$

Finally we have,

$$\begin{aligned} d_H\left(T[A],\ T[B]\right) &= \max\left\{ d_2(T[A],T[B]),\ d_2(T[B],T[A]) \right\} \\ &\le \max\left\{ c\,d_2(A,B),\ c\,d_2(B,A) \right\} \\ &= c\max\left\{ d_2(A,B),\ d_2(B,A) \right\} \\ &= c\,d_H(A,B) \end{aligned}$$

Hence, T is a contraction map under d_H. ■

Theorem 3.6 : If T_1 and T_2 are two contraction mappings on a metric space (X, d) with contraction factor c_1 and c_2, then $T: H(X)\to H(X)$ defined by $T(A)=T_1(A)\cup T_2(A)\quad \forall\ A\in H(X)$ is also a contraction map with contraction factor $\max\{c_1, c_2\}$.

Proof: We first prove the following lemma,

Lemma 3.6.1: For any sets $A, B, C, D \in H(X)$,

$$d_H(A\cup B,\ C\cup D) \le \max\{d_H(A,C),\ d_H(B,D)\}.$$

Proof: First we prove that $d_2(A\cup B, C) = \max\{ d_2(A,C),\ d_2(B,C)\}$.

Consider the point $a\in A$ such that $d_1(a,C)=\max\{d_1(x,C): x\in A\}$ and the point $b\in B$ such that $d_1(b,C)=\max\{d_1(y,C): y\in B\}$. Now,

$$d_2(A\cup B, C) = \max\{ d_1(a,C),\ d_1(b,C)\} = \max\{d_2(A,C),\ d_2(B,C)\}$$

Now for any $x\in A$ we have,

$$d_1(x, B\cup C) = \min\{d(x,\alpha): \alpha\in B \text{ or } \alpha\in C\}$$

Therefore, $d_1(x, B\cup C) = \min\{d(x,\alpha): \alpha\in B\}$ or $d_1(x, B\cup C) = \min\{d(x,\alpha): \alpha\in C\}$

i.e., $d_1(x, B\cup C) = d_1(x,B)$ or $d_1(x, B\cup C) = d_1(x,C)$

Thus if x is a point on A farthest from $B\cup C$, we have

$$d_2(A, B\cup C) = d_1(x, B\cup C) = \min\{d_1(x,B),\ d_1(x,C)\} \le \min\{d_2(A,B),\ d_2(A,C)\}$$

Now,

$$d_H(A\cup B,\ C\cup D) = \max\{d_2(A\cup B,\ C\cup D),\ d_2(C\cup D, A\cup B)\}$$

$$\begin{aligned} \Rightarrow d_H(A\cup B,\ C\cup D) &= \max\{d_2(A, C\cup D),\ d_2(B, C\cup D), d_2(C, A\cup B), d_2(D, A\cup B)\} \\ &\le \max\{d_2(A,C),\ d_2(B,D),\ d_2(C,A), d_2(D,B)\} \\ &= \max\{\max\{d_2(A,C),\ d_2(C,A)\}, \max\{d_2(B,D), d_2(D,B)\}\} \\ &= \max\{d_H(A,C),\ d_H(B,D)\} \end{aligned}$$

Proof of the main theorem: Let $A, B\in H(X)$ and $\alpha = \max\{\alpha_1, \alpha_2\}$ where α_1 and α_2 are contraction factors of T_1 and T_2 respectively.

Now,

$$\begin{aligned} d_H\left((T_1\cup T_2)(A), ((T_1\cup T_2)(B))\right) &= d_H\left(T_1(A)\cup T_2(A),\ T_1(B)\cup T_2(B)\right) \\ &\le \max\{ d_H(T_1(A), T_1(B)),\ d_H(T_2(A), T_2(B))\} \\ &\le \max\{\alpha_1 d_H(A,B),\ \alpha_2 d_H(A,B)\} \\ &\le \max\{\alpha\, d_H(A,B),\ \alpha\, d_H(A,B)\} \\ &= \alpha\, d_H(A,B) \end{aligned}$$

Therefore, $T_1\cup T_2$ is a contraction mapping with contraction factor $\alpha = \max\{\alpha_1, \alpha_2\}$. ■

4. Construction of fractal objects by Iterated Functions System:

After setting the stage for the space in which fractal objects will live, we now discuss the effect of a finite set of contraction mappings acting iteratively on a point of a complete metric space. Barnsley defines a iterated function system [3] in the following way:

Definition 4.1: Let $T_1, T_2, \cdots, T_n \quad (n \geq 2)$ be a finite collection of contraction mappings defined on a complete metric space (X, d) . Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$ be their respective contraction factors. Then the system $(X; T_i : i = 1, 2, \cdots, n)$ is called a *iterated functions system* (abbreviated as *"IFS"*) with contraction factor $\alpha = \max\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

Theorem 4.1: An IFS $(X; T_i : i = 1, 2, \cdots, n)$ has a unique fixed point $L \in H(X)$ such that $L = \bigcup_{i=1}^{n} T_i(L)$. Also,

$$L = \bigcup_{i=1}^{n} T_i(M), \; for \;\; any \;\; M \in H(X).$$

Proof: By theorem 3.5 each $T_i \; (i=1,2, \cdots, n)$ is a contraction mapping on the Hausdorff space $H(X)$ with contraction factor α_i , on effect of which each compact subset of X transform to another compact subset of X . Again, union of any two contraction mapping is also a contraction mapping (Theorem 3.6) , the operator $T = \bigcup_{i=1}^{n} T_i : H(X) \to H(X)$ defined by

$$T(A) = \bigcup_{i=1}^{n} T_i(A) \quad \forall \; A \in H(X)$$

is also a contraction mapping with contraction factor $\alpha = \max\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. As a consequence of theorem 3.4, this contraction mapping T has a unique fixed point, say L i.e., $T(L) = \bigcup_{i=1}^{n} T_i(L) = L$ which satisfies $L = \bigcup_{i=1}^{n} T_i(M), \; for \;\; any \;\; M \in H(X).$

Definition 4.2[3]: If $(X; T_i : i = 1, 2, \cdots, n)$ is a IFS, then the operator $T = \bigcup_{i=1}^{n} T_i : H(X) \to H(X)$, defined by $T(A) = \bigcup_{i=1}^{n} T_i(A) \quad \forall \; A \in H(X)$ has a unique fixed point, say L. This fixed point $'L'$ is called the *attractor* of the IFS $(X; T_i : i = 1, 2, \cdots, n)$.

Clearly each $T_i \; (i = 1, 2, \cdots, n)$ transforms members of $H(X)$ into geometrically similar sets. The attractor of such a collection $T_i's$ must be a self-similar set, being a number of smaller similar copies of itself. Hence, the fundamental property of an IFS is that it determines a unique attractor, which usually is a fractal. Based on these knowledge we now discuss the mathematical development to provide two algorithms for constructing pictures of attractors of an IFS.

4.3 Algorithm 1(The Deterministic Algorithm):

i. Consider the IFS $(X; T_i : i = 1, 2, \cdots, n)$.

ii. Choose a compact set $A_0 \in R^2$.

iii. Compute successively the sequence $\{A_i : i=1,2, \cdots n\}$ with the role

$$A_{i+1} = \bigcup_{j=1}^{n} T_j(A_i) \quad ; \; i = 0,1,2, \cdots$$

iv. Plot the sequence $\{A_i : i=1,2, \cdots n\} \subset H(X)$ successively on the computer screen.

This sequence of points converses to the attractor of the IFS in Hausdorff metric which is very often a fractal set.

We illustrate this algorithm by considering the IFS $(R^2; T_1 , T_2)$ where, $T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and $T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ 0.5 & -0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We can write this two affine transformations in tabular form of IFS codes as:

Affine Transformations	Rotation / Dilation				Translation	
	a	b	c	d	e	f
T_1	0.5	-0.5	0.5	0.5	0	0
T_2	-0.5	-0.5	0.5	-0.5	1	0

Table 4.1 [*IFS codes for* $(R^2 : T_1 , T_2)$]

Considering the initial set A_0 as the straight line segment of unit length, the images constructed by the first four iterations are given in the *figure 4.1* and its attractor is given in *figure 4.2* which is the famous Harter-*Heighway Dragon* curve named after NASA physicists William Harter and John Heighway.

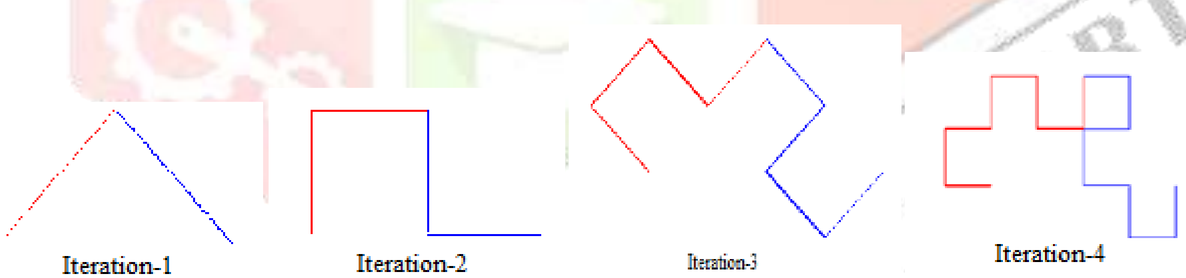


Figure 4.1 [*First four iteration of the IFS* $(R^2 : T_1 , T_2)$]

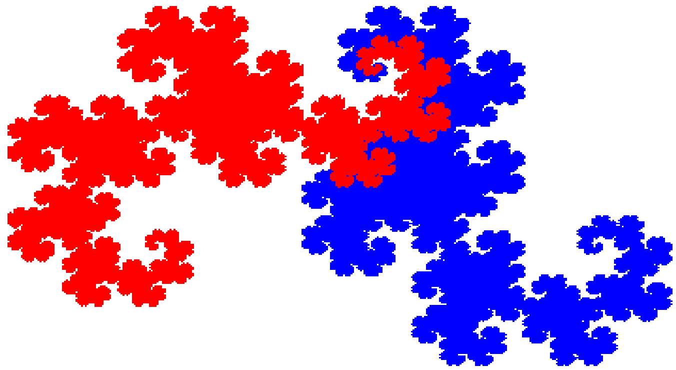


Figure 4.2 [*Attractor of the IFS* $[\left(R^2 : T_1, T_2\right)]$

Some other IFS codes and their respective attractors produced by this algorithm are given below:

a) IFS codes for fractal 'K':

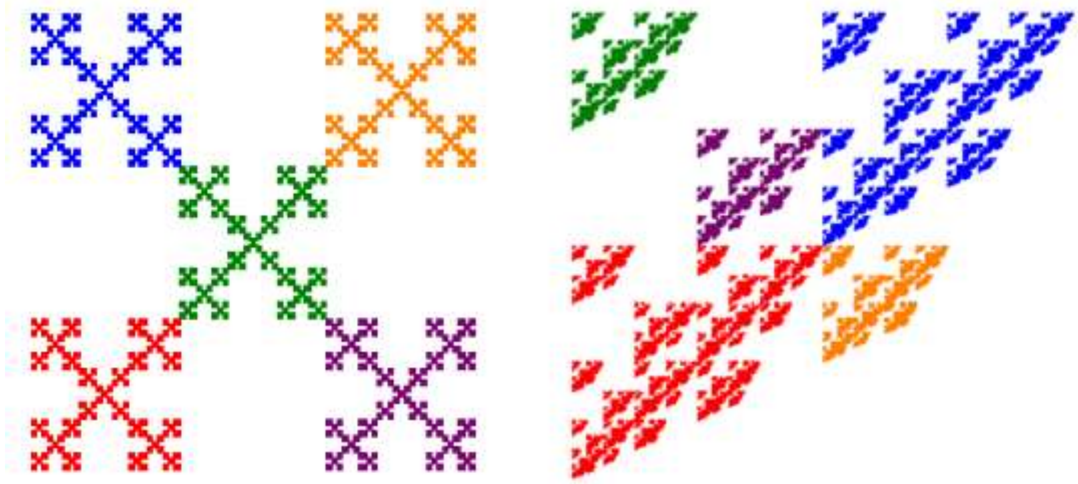
Affine Transformations	Rotation / Dilation				Translation	
	a	b	c	d	e	f
f_1	0.333	0	0	0.333	0	0
f_2	0.333	0	0	0.333	0	0.667
f_3	0.333	0	0	0.333	0.667	0.667
f_4	0.333	0	0	0.333	0.667	0
f_5	0.333	0	0	0.333	0.333	0.333

Table 4.2

b) Fractal 'G':

Affine Transformations	Rotation / Dilation				Translation	
	a	b	c	d	e	f
g_1	0.5	0	0	0.5	0	0
g_2	0.25	0	0	0.25	0	0.75
g_3	0.25	0	0	0.25	0.25	0.5
g_4	0.25	0	0	0.25	0.5	0.25
g_5	0.5	0	0	0.5	0.5	0.5

Table 4.3



(a) Fractal 'K' (b) Fractal 'G'

Figure 4.3 [Attractors of the IFS given in (a) Table 4.2, (b) Table 4.3]

4.4 Algorithm 2(The Random Iteration Algorithm):

In this algorithm, instead of choosing a compact set, we choose a singleton set or a point (usually the origin). Form this point new points are iteratively computed by randomly applying one of the transformations involved in the IFS. Ignoring the issue of how the attractor is formed by discarding first few iterations (in most of the practical purposes 50 iterations are sufficient) this algorithm reduces the computational time and memory cells of the computer involved in this process. Also in deterministic algorithm each of the transformation contributes equal points in the attractor set, but in most of the natural fractals we need different number of points for different parts of the attractor. The random iteration algorithm suitably handled this problem by assigning probability density function p_i to each transformation T_i involved in the IFS such that $\sum_{i=1}^{n} p_i = 1$.

i. Consider the IFS $\left(X; T_i : i = 1, 2, \cdots, n \right)$.

ii. Assign probability density function p_i to each transformation T_i involved in the IFS such that $\sum_{i=1}^{n} p_i = 1$.

iii. Start with an arbitrary point in the plane (usually the origin).

iv. Randomly choose a transformation T_i according to the probability p_i.

v. Transform the point using T_i and plot it.

vi. Go to step iv.

vii. Continue the process for some pre-defined number of iterations.

Below we have given some fractals together with their IFS codes produce by using this random iteration algorithm:

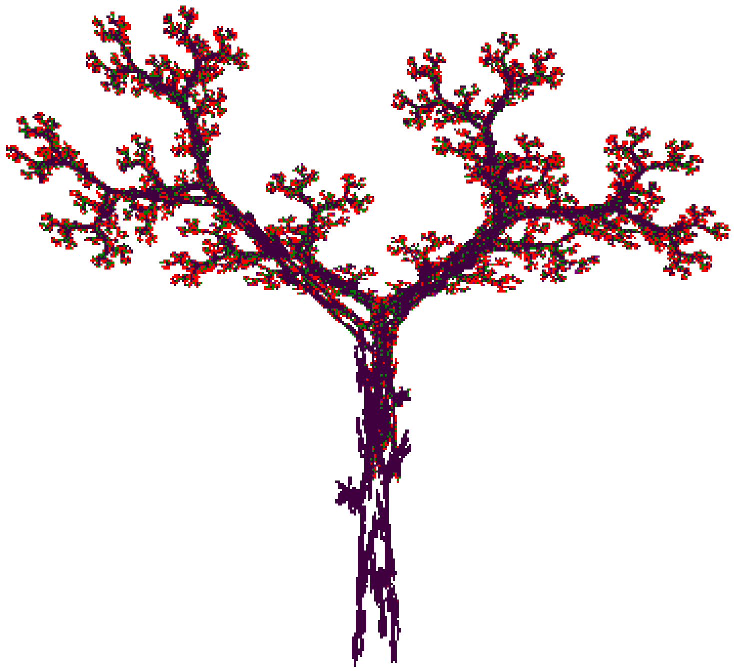


Figure 4.4 [*A Bloo* min *g Tree*]

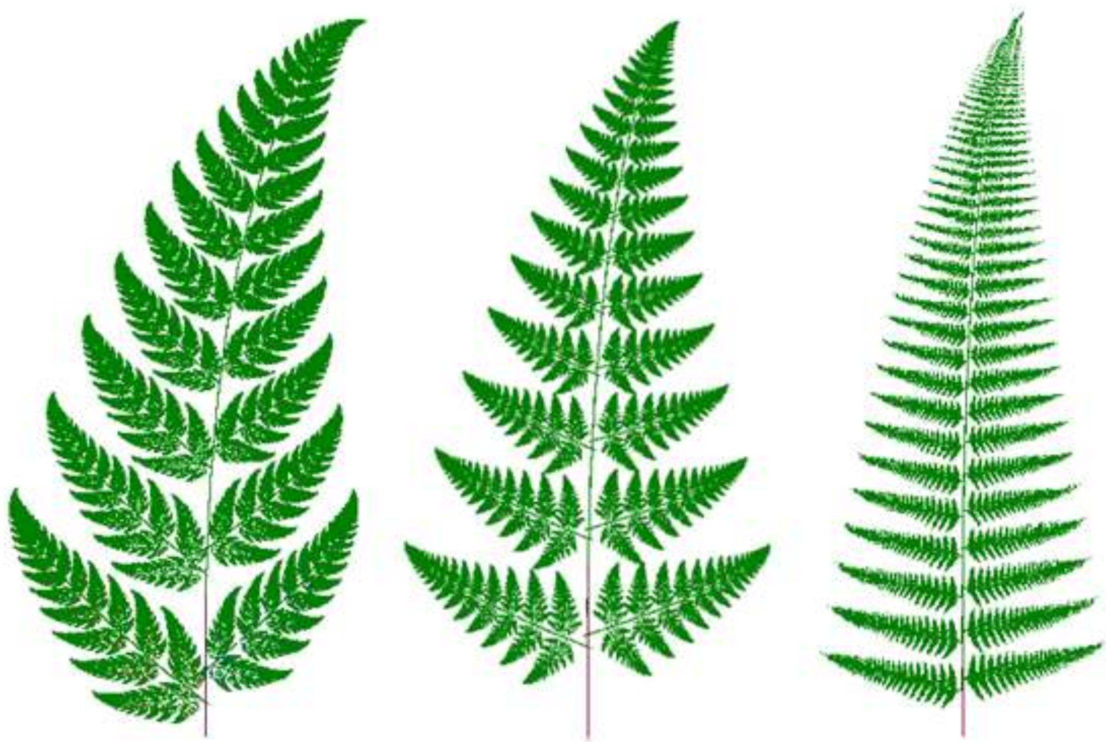


Figure 4.5 [*Three Mutant Varieties of Fern*]

IFS codes for fractal given in the figure 4.4:

Affine Transformations	Rotation / Dilation				Translation		Probability
	a	b	c	d	e	f	p_i
T_1	– 0.637	0	0	0.501	0.866	0.25	0.13
T_2	0.195	– 0.488	0.344	0.443	0.443	0.245	0.25
T_3	0.462	0.414	– 0.252	0.361	0.251	0.569	0.25
T_4	- 0.058	– 0.07	0.453	– 0.111	0.598	0.097	0.25
T_5	– 0.035	0.07	- 0 .469	– 0.022	0.488	0.507	0.12

Table 4.4 [*IFS codes for 'Blooming Tree'*]

IFS codes for fractal given in the figure 4.5:

Affine Transformations	Rotation / Dilation				Translation		Probability
	a	b	c	d	e	f	p_i
T_1	0	0	0	0.16	0	0	0.01
T_2	0.85	0.04	– 0.04	0.85	0	1.6	0.85
T_3	0.2	– 0.26	0.23	0.22	0	1.6	0.07
T_4	- 0.15	0.28	0.26	0.24	0	0.4	0.07

Table 4.5 [*IFS codes for 'Fern'*]

5. Computation of Fractal Dimension:

One of the typical characteristics of fractals is its dimension, which is essentially a measure of self-similarity. It is sometimes referred as similarity dimension. There are a number of non-equivalent ways of finding fractal dimension. One of the most theoretically strong concept of fractal dimension is Hausdorff dimension . But in most of the practical purposes it is difficult to calculate. One of the advantages of using an iterated function system is that the Hausdorff dimension of the attractor is often relatively easy to calculate in terms of the defining contraction factors. Before begin to exploit concept of IFS to find Haudorff dimension of a fractal set we first define the definitions of Hausdorff dimension.

5.2. Hausdorff Dimension[4] : Let (X,d) be a metric space, $E \subseteq X$. The diameter of the set E , dented by $|E|$, and is defined as

$$|E| = \sup\{d(x,y): x, y \in X\}$$

Now for each positive number ε, Consider the possible countable coverings $B = \{B_i\}$ of E $\left(i.e.\ E \subset \bigcup_{i=1}^{n} B_i \right)$ such that $|B_i| < \varepsilon$, $\forall\ i$.

Now for each non-negative real number k, we consider the sum $m_k(B)$ of all k^{th} powers of $|B_i|$ i.e. $m_k(B) = \sum_i |B_i|^k$, where this sum might be infinite. An attempt to minimize this sum we define, $m_k^{\varepsilon}(E) = \inf\ \{m_k(B)\} = \text{glb}\left\{ |B_i|^k : |B_i| < \varepsilon, E \subset \bigcup_i B_i \right\}$, where this greatest lower bound is taken over all possible countable covers of E by sets of diameter at most ε. Note that, when some number of elements from a set is deleted the infimum of the set may be increase, so as we decrease the value of ε, the class of such coverings of E diminishes and hence the value of $m_k^{\varepsilon}(E)$ increases. Thus the sequence of greatest lower bounds converges, i.e. $\lim_{\varepsilon \to 0} m_k^{\varepsilon}(E)$ exist. Note that this limit may take the value infinite. we denote this limit as $M_k(E)$.

Lemma 5.2.1: If $M_k(E) < \infty$ and $k < t$, then $M_t(E) = 0$.

Proof: Let $B = \{B_i\}$ be a countable cover of the set E with sets of diameter at most ε. Then we have,

$$m_t^{\varepsilon}(E) = \inf\ \{m_t(B)\} \le \sum_i |B_i|^t \le \varepsilon^{t-k} \sum_i |B_i|^k$$

Considering all such coverings of E and then taking their greatest lower bound we obtain,

$$m_t^{\varepsilon}(E) \le \varepsilon^{t-k} m_k^{\varepsilon}(E)$$

Now taking limit as $\varepsilon \to 0$, we obtain $M_t(E) = 0$. ∎

An immediate consequence of this lemma is that there exist a non-negative number $d_H(E)$ of k where the value of $M_k(E)$ jumps from infinity to zero, i.e,

$$M_k(E) = \begin{cases} \infty, & if\ \ k < d_H(E) \\ 0, & if\ \ k > d_H(E) \end{cases}$$

This number $d_H(E)$ is called the *Hausdorff dimension* of the set E .

The Hausdorff dimension of a self-similar fractal set constructed by an IFS as attractor can be easily calculated if the fractal satisfies a certain condition called open set condition.

Definition 5.3[5]: Consider the IFS $\left(X;T_i:i=1,2,\cdots,n\right)$. Now each T_i of this IFS has the *open set condition* if there exist a non-empty, bounded open set U such that

i. $T_i\left(U\right)\subset U,\ \forall i=1,2,\ \cdots,n$
ii. $T_i\left(U\right)\cap T_j\left(U\right)=\phi,\ \forall\ i\neq j$.

Theorem 5.1: Let $\{c_i: i=1,2,\ \cdots,n\}$ be the contraction factors of the affine transformations involved in the IFS $\left(X;T_i:i=1,2,\cdots,n\right)$ whose attractor is the fractal set A, then $d_H(A)=d$ where d is the solution to the equation:

$$\sum_{i=1}^{n} c_i{}^d = 1.$$

The proof of this theorem can be found in [3].

The following examples clarify the method of finding Haudorff dimensions of fractals produced by an IFS using Theorem 5.1:

Consider the two fractals given in the *figure 4.3(a) & (b)*. The IFS of both of these fractal consists of five contraction mappings. By considering the set U as the open square connecting the points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ it is easy to see that these two sets of contraction mappings satisfies the open set condition.

Now all the contraction mappings of the IFS of fractal 'K' (given in Table 4.2) are of contraction factor $\frac{1}{3}$.

Hence by Theorem 5.1, its Haudorff dimension d is given by the equation:

$$\sum_{i=1}^{5}(c_i)^d = 1 \quad\Rightarrow\quad 5\left(\frac{1}{3}\right)^d = 1 \quad\Rightarrow\quad d = \frac{\log 5}{\log 3}$$

Again the contraction mappings involved in the fractal 'G' (given in Table 4.3) are of contraction factors $\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2}$ respectively. Therefore, the Haudorff dimension d of the fractal 'G' satisfies the equation,

$$\left(\frac{1}{2}\right)^d+\left(\frac{1}{4}\right)^d+\left(\frac{1}{4}\right)^d+\left(\frac{1}{4}\right)^d+\left(\frac{1}{2}\right)^d = 1 \qquad \ldots \qquad (5.1)$$

Letting $x = 2^d$, this equation gives,

$$\frac{2}{x}+\frac{3}{x^2} = 1 \Rightarrow x=-1,\ or\ \ x=3$$

As $x = 2^d \Rightarrow x > 0$, so we must have $x = 3$, and so $3 = 2^d \Rightarrow d = \frac{\log 3}{\log 2}$.

Hence, $d_H(G)=\frac{\log 3}{\log 2}$.

6.Conclusion: One main idea of IFS is that it formally encodes the idea of self-similarity which is one of the prime characteristics of a fractal object. Here two algorithms were presented. The deterministic algorithm was a direct consequence of the theorem 4.1. Another simpler way of constructing the attractor is random iteration algorithm, which takes as argument one point $x_0 \in X$, a set of contraction mappings $\{T_i : i = 1, 2, \cdots, n\}$, together with an probability set $\{p_1, p_2, \cdots, p_n\}$ such that $\sum_{i=1}^{n} p_i = 1$ and then generate a sequence of points $\{x_n\}$ where $x_n = T_i(x_{n-1})$, here p_i acts as a probability of choosing T_i. As $n \to \infty$, this set of points clusters around a set called the attractor of the IFS. As this set is self similar, this set is often a fractal set. Note that, since the selection of the map T_i for generating the point $x_n = T_i(x_{n-1})$ is random, in some initial stage of the construction process the attractor might not be the same with the attractor which we generated earlier with same IFS codes. However, in most of the natural fractals it is highly acceptable as if we more interested in a certain region of the attractor, we may assigned higher probability to the mapping which maps points in that region.

References:

[1]. Ambum,P.,Grant,E.,Whitted,T.;"*Managing geometric complexity with enhanced procedural methods*"; Computer Graphics; 20,4(1986)

[2]. Barnsley,M.F.,Demko,S.;"*Iterated function systems and the global construction of fractals*";Proceedings of the Royal Society of London; Series A,399(1985).

[3]. Barnsley,M.F.;"*Fractals Everywhere*"; 2nd ed.; Academic Press, San Diego New York; ISBN-10:0-12-079069-6.

[4]. Beardon, A.F., "*Iteration of Rational Functions*" , Springer-Verlag, Berlin/ Heidelberg, 1991.

[5]. DeLorto, R., "*Fractal dimensions and Julia sets*", EWU Masters Thesis Collection, Easterns Washington University, http://dc.ewu.edu/theses, 2013.

[6]. Demko,S.,Hodges,L.,Naylor,B.;"*Construction of fractal objects with iterated function systems*";Computer Graphics; 19,3;1985.

[7]. Dutil,N.;"*Construction of Fractal Object with Iterated Function systems*"; www.cs.mcgill.ca/~nduti/project.pdf.

[8]. Falconer. K (2003):"*Fractal Geometry: Mathematical Foundation and Applications*" Second Edition, John Willey and Sons Ltd., USA.

[9]. Fatou,P.;"*Sur les equations fonctionelles*", Bul. Society of Mathematics, Fr.47(1919),48(1920)

[10]. Feldman,D.P.; "*Chaos and Fractals: An Elementary Introduction*"; Oxford University press; ISBN 978--0-19-956644-0; 2014.

[11]. Fournier,A.,Fussel,D. and Carpenter,L.; "*Computer rendering of stochastic models*"; Comm. of the ACM; 25 (1982).

[12]. Hutchinson,J.E.;"*Fractals and self-similarity*"; Indiana University Journal of Mathematics; 30,5(1981).

[13]. Kawaguchi,Y.;"*A morphological study of the form of nature*"; Computer Graphics; 16,3(1982).

[14]. Mandelbrot,B.B.;"*Fractals: Form, Chance and Dimension*"; San Francisco: W.H.Freeman and Co.,1977.

[15]. Mandelbrot,B.B.;"*The Fractal Geometry of Nature*"; San Francisco: W.H.Freeman and Co.,1983.

[16]. Mandelbrot,B.B:"Comment on computer rendering of fractal stochastic models"; Comm. of the ACM 25,8 (1982).

[17]. Miller. G.S.P.;"*The definition and rendering of terrain maps*"; Computer Graphics; 20,4(1956).

[18]. Oppenheimer, P.E.; "*Real time design and animation of fractal plants and trees*"; Computer Graphics; 20,4(1986).

[19]. Ott, Edward; " *Chaos in dynamical systems*"; Cambridge University press, 0 521 43215 4; 1993.

[20]. Simmon,G.F.;"*Introduction to Topology and Modern Analysis*"; McGraw-Hill Book Co.; ISBN 0-07-Y85695-8(1963).

[21]. Smith,A.R.;"*Plants, fractals, and formal languages*"; Computer Graphics; 18,3(1984)