Solution of Fractional Ordinary Differential Equation by MAHGOUB Transform

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***Abstract**:* In this paper a method is introduced to find the solution of fractional ordinary equation by Mahgoub Transform . We give some basic properties like derivative, periodic function by the Mahgoub Transform.

Key words: Mahgoub transform, fractional derivative.

1. INTRODUCTION:

The solution for non linear fraction differential equation now span a half century or more and play a crucial role in several theoretical and applied sciences such as , but certainly not limited to, theoretical biology and ecology , solid state physics , viscoelasticity , fibber optics , signal processing and electric control theory , stochastic based finance , and thermodynamics

In this work our aim is to exhibit exact solution of some homogeneous and non-homogeneous fractional ordinary differential equation by using the Mahgoub Transform method. We apply the Mahgoub Transform method to obtain new exact solution of both homogeneous and non-homogeneous fractional ordinary differential equation.

2. Fractional calculus and Mahgoub Transform:

The theory of fractional calculus is important role in many field of applied and pure mathematics. The association of differential integral transform with fractional integers and derivatives are used to solve different types of differential and integral equations. A derivative of fractional order, in the Abel – Riemann scene [2013] is define by

$$D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\frac{d}{dt^m}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau, & m-1<\alpha\le m \\ \frac{d^m}{dt^m}f(t) & ,\alpha = m \end{cases} \quad (1)$$

Where m Є Z^+ and the integral operator is defined by implementing an integral of fractional order in Abel – Riemann sense as

$$D^{-\alpha}[f(t)] = j^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(t-\tau)d\tau \quad ,t>0,\alpha>0 \quad (2)$$

Podlubny [1999] introduced the fundamental properties of fractional integration and differentiation respectively , described as

$$j^{\alpha}[t^n] = \Gamma t^{n+\alpha}$$

$$D^{\alpha}[t^n] = \frac{\Gamma(1+n)}{\Gamma(1-n+\alpha)}t^{n-\alpha}$$

The caputoderivative is defined by

$$D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha \le m \\ \frac{d^m}{d\beta^m} f(t) & ,\alpha = m \end{cases}$$

$$j^{\alpha}[D^{\alpha}[f(t)]] = f(t) - \sum_{k=0}^{\infty} f^k(0) \frac{t^k}{k!} \quad (3)$$

2.1 MAHGOUB TRANSFORM

we can take set A the function is defined in the form

$$A = \left\{ f : \left| f(t) < pe^{\frac{|t|}{ki}} if t \in (-1)^i \times [0, \infty), i = 1,2; \varepsilon_i > 0 \right. \right\} \quad (4)$$

The constant P must be finite number $\varepsilon_1 \& \varepsilon_1$ may be finite or infinite.

Then the Mahgoub Transform define as,

$$M\left(f(t)\right) = H(v) = v\int_0^{\infty} f(t) e^{-vt} dt, \;\; t \ge 0, \;\; \varepsilon_1 \le v \ge \varepsilon_2 \quad (5)$$

2.2 Mahgoub Transform Of Derivative

Let function f(t) then derivative of $f(t)$ with respect to t and the n^{th} order derivative of the same with respect to t are respectively. Then Mahgoub transform of derivative given by

$$M\,[f^n(t)] = v^n H(v) - \sum_{k=0}^{n-1} v^{n-k} f^k(0) \quad (6)$$

For n = 1, 2,3 ….. in equation (6) give Mahgoub Transform of first and second derivative of f(t) w.r.t. t

$$M\left[f'(t)\right] = vH(v) - vf(0)$$

$$M\left[f''(t)\right] = v^2 H(v) - vf'(0) - v^2 f(0)$$

2.3 Convolution theorem of Mahgoub Transform:

Let f(t) and g(t) are two function then Mahgoub Transform of convolution theorem of two function is given by

$$M\,(f * g) = \frac{1}{v} M(f) M(g)$$

2.4 Theorm: The Magoub transform of a piece wise periodic function f(t) with period p is

$$M\,\{f(t)\} = \frac{v}{1-e^{-vp}} \int_0^p e^{-tv} f(t) dt; \qquad v > 0$$

Proof:

Let Function f(t) is said to be periodic function T > 0 if

$f(t) = f(T+t) = f(2T+t) = f(nT+t)$

By definition

$$M\,[f(t)] = v\int_0^{\infty} f(t) e^{-vt}\, dt$$

$$M\,[f(t)] = v\left[\int_0^p f(t)e^{-vt}\, dt + \int_p^{2p} f(t)e^{-vt}\, dt + \cdots \int_{np}^{\infty} f(t)e^{-vt}\, dt\right] \quad (7)$$

Put, t = u+p in second integral and up to

.

.

$t^n = u + (n-1)p \; in \; n^{th}$ integral in equation (7), Now we get new limit for each integral are 0 to p and equation (7) by periodicity

$f(t+p) = f(t)\,, f(t+2p) = f(t),$ and so on therefore

$M\{f(t)\} = v\big[f(u)e^{-uv} + \int_0^p f(u)e^{-v(u+p)}du + \int_0^p f(u)e^{-v(u+2p)}du + \cdots\big]$

$M\{f(t)\} = v\int_0^p f(u)\,[1 + e^{-vp} + e^{-2vp} + \cdots]e^{-uv}du$

$M\{f(t)\} = v.\frac{1}{1-e^{-vp}}\int_0^p f(u)e^{-uv}du$

$$M\{f(t)\} = \frac{v}{1-e^{-vp}}\int_0^p e^{-tv}f(t)dt, \qquad v > 0 \qquad (8)$$

The Mahgoub transform of periodic function f(t) of period p is obtain by integrating $e^{-tv}\,f(t)$ in interval (0,p) with respect to t and multiplying the resultant by the factor $(1-e^{-vp})^{-1}$.

Proposition 1: If H(v) is the Mahgoub transform of the function f(t) then the Mahgoub transform of fraction integral of order $\propto$ is defined by [1]

$$M\left[D^{-\propto}[f(t)]\right] = \frac{(\propto-1)!}{\Gamma(\propto)}H(v)V^{\propto} \qquad (9)$$

Proof:

Applying Mahgoub transform in the equation (9), we have

$M\left[D^{-\propto}[f(t)]\right] = \frac{1}{\Gamma(\propto)}M[t^{\propto-1}] * M[f(\tau)],$

$$= \frac{(\propto-1)!}{\Gamma(\propto)}H(v)V^{\propto} \qquad (10)$$

Proposition 2: If H(v) is the Mahgoub transform of the function f(t) then the Mahgoub transform of fractional derivative of order $\propto$is defined as

$M\,[f^{\propto}t] = V^{\propto}H(v) - \sum_{k=0}^{\propto-1}V^{\propto-k}f^{k}(0),$

Fractional derivative

$$V^{\propto}\left[H(v) - \sum_{k=1}^{n}V^{\propto+k}\left(D^{\propto-k}f(t)\right)_{t=0}\right] \qquad (11)$$

3. Application of Mahgoub transform: In this section, we discuss the solution of fractional ordinary differential equation using general properties with initial condition.

3.1 Solve the non-homogeneous fractional ordinary differential equation as,

$[D^{\alpha}u(t)] + [D^2u(t)] + [Du(t)] + [u(t)] = d$

With initial condition, y(0) =f(0)

Solution:

Given equation;

$[D^{\alpha}u(t)] + [D^2u(t)] + [Du(t)] + [u(t)] = d$

Apply the Mahgoub transform both sides we obtain

$M\,[D^{\alpha}u(t)] + M\,[D^2u(t)] + M\,[Du(t)] + M\,[u(t)] = M\,[d]$

$v^{\alpha}\left[H(v) - \sum_{k=1}^{n}v^{\alpha+k}\left(D^{\alpha-k}f(t)\right)_{t=0}\right] + v^2H(v) - vf'(0) - v^2f(0) + v(H)(v) - vf(0) + H(v) = M\,[d]$

$$H(v)[v^{\alpha}+v^2+vH]=v^{\alpha}\sum_{k=1}^{n}v^{\alpha+k}\left(D^{\alpha-k}f(t)\right)_{t=0}+vf'(0)+v^2f(0)+vf(0)+1$$

Taking the inverse Mahgoub Transform both side then we get

$$M^{-1}[H(v)]=M^{-1}\left[\frac{v^{\alpha}\sum_{k=1}^{n}v^{\alpha+k}\left(D^{\alpha-k}f(t)\right)_{t=0}+vf'(0)+v^2f(0)+vf(0)+1}{H(v)[v^{\alpha}+v^2+vH]}\right]$$

Now apply the initial condition, we get exact solution ;

$$y(t)=M^{-1}\left[\frac{v^{\alpha}\sum_{k=1}^{n}v^{\alpha+k}\left(D^{\alpha-k}f(t)\right)_{t=0}+vf'(0)+v^2f(0)+vf(0)+1}{H(v)[v^{\alpha}+v^2+vH]}\right]$$

Solve the equation and find out the existing solution f(t).

3.2 Solve the non-homogenious fractional ordinary differential equation with initial condition.

$$D^{0.5}[u(t)+u(t)]=t^2-\frac{\Gamma(3)}{\Gamma(2.5)}t^{1.5} \qquad t>0$$

With initial condition, $[D^{-0.5}u(t)]_{t=0}=0$

Solution:

Given equation

$$D^{0.5}[u(t)]+[u(t)]=t^2-\frac{\Gamma(3)}{\Gamma(2.5)}t^{1.5} \qquad t>0$$

$$And, \qquad [D^{-0.5}u(t)]_{t=0}=0$$

We Applying the Maughoub transform both side

$$M\,[D^{0.5}u(t)]+M\,[u(t)]=M\,(t^2)-\frac{\Gamma(3)}{\Gamma(2.5)}M\,[t^{1.5}]$$

$$v^{0.5}\,H(V)-\sum_{k=1}^{0.5}v^{0.5-k}\,[D^{0.5-k}u(t)]_{t=0}+H(V)=\frac{2!}{V^2}-\frac{\Gamma(3)}{\Gamma(2.5)}\frac{(1.5)!}{V^{1.5}}$$

$$if\ k=1\ then,$$

$$v^{0.5}\,H(V)-v^{0.5-1}[D^{0.5-1}u(t)]_{t=0}+H(V)=\frac{2!}{V^2}-\frac{\Gamma(3)}{\Gamma(2.5)}\frac{(1.5)!}{V^{1.5}}$$

$$v^{0.5}\,H(V)-v^{-0.5}[D^{-0.5}u(t)]_{t=0}+H(V)=\frac{2!}{V^2}-\frac{\Gamma(3)}{\Gamma(2.5)}\frac{(1.5)!}{V^{1.5}}$$

We Applying the initial condition,

$$H(V)[V^{0.5}+1]=\frac{2!}{V^2}+2\frac{(1.5)!}{V^{1.5}}$$

$$H(V)[V^{0.5}+1]=\frac{2!}{V^2}+2\frac{1}{V^{1.5}}$$

$$H(V)[V^{0.5}+1]=\frac{2!}{V^2}\Big[[V^{0.5}+1]\Big]$$

$$H(V) = \frac{2!}{V^2}$$

Taking the inverse Mahgoub Transform both side for the value of y(t),

$$M^{-1}[H(v)] = M^{-1}\left[\frac{2!}{v^2}\right]$$

We get exact solution by the Mahgoub Transform method as follows:

$$y(t) = t^2$$

3.3 Solve the non-homogenious fractional ordinary differential equation.

$$D^{\frac{1}{2}}\, y(t) + y(t) = \frac{1}{2} + t^{1/2}$$

With initial condition, $\left[D^{-\frac{1}{2}}y(t)\right]_{t=0} = 0$

Solution:

Given equation

$$D^{\frac{1}{2}}\, y(t) + y(t) = \frac{1}{2} + t^{1/2}$$

With initial condition, $\left[D^{-\frac{1}{2}}y(t)\right]_{t=0} = 0$

We Applying the Maughoub transform both side,

$$M\left[D^{\frac{1}{2}}y(t)\right] + M\,[y(t)] = M\left(t^{1/2} + \frac{1}{2}\right)$$

$$V^{\frac{1}{2}}H(V) - \sum_{k=1}^{\frac{1}{2}} V^{\frac{1}{2}-k}\left[D^{\frac{1}{2}-k}y(t)\right]_{t=0} + H(V) = \frac{\frac{1}{2!}}{v^{\frac{1}{2}}} + \frac{1}{2}$$

Taking k = 1 then,

$$V^{\frac{1}{2}}H(V) - V^{\frac{1}{2}-1}\left[D^{\frac{1}{2}-1}y(t)\right]_{t=0} + H(V) = \frac{\frac{1}{2}}{v^{\frac{1}{2}}} + \frac{1}{2}$$

$$V^{\frac{1}{2}}H(V) - V^{-\frac{1}{2}}\left[D^{-\frac{1}{2}}y(t)\right]_{t=0} + H(V) = \frac{\frac{1}{2}}{v^{\frac{1}{2}}} + \frac{1}{2}$$

Using the initial condition

$$H(V)\left[V^{\frac{1}{2}} + 1\right] = \frac{1}{2}\frac{1}{v^{\frac{1}{2}}}\left[V^{\frac{1}{2}} + 1\right]$$

$$H(V) = \frac{1}{2}\ \frac{1}{v^{\frac{1}{2}}}$$

Taking the inverse Mahgoub Transform both side for the value of y(t)

$$M^{-1}[H(V)] = M^{-1}\left[\frac{1}{2}\ \frac{1}{v^{\frac{1}{2}}}\right]$$

We get exact solution by the Mahgoub Transform method follows:

$$y(t) = t^{1/2}$$

4.Discussion and conclusion:

We have applied Mahgoub Transform for Fractional ordinary differential equation as well as periodic function. It is found that the Mahgoub Transform has an extensive affinity with the solutions of differential and integral equations, and more specifically with the Fractional differential equations which has been the centre forum of this paper. We found that the solution of fraction ordinary differential equation can be obtained in the form of distribution fractional ordinary differential equations when distributed natural transform are invoked.

5.Refenrence

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