ANALYTIC CONTINUATION AND GERM TOPOLOGY OF MEROMORPHIC FUNCTION

P.PRIYANKA [1] S.ASWINI[2]
HOD & ASSISTANT PROFESSOR[1] & PG [2]
ADHIYAMAN ARTS & SCIENCE COLLEGE FOR WOMEN , UTHANGARAI

INTRODUCTION

In complex analysis, branch of mathematics,analytic continuation is a technique to extend the domain of a given analytic function. Analytic continuation often succeeds in defining further values of a function .In mathematics, the notion of a germ of an object in/on a topology space is an equivalence class of that object and others of the same kind which captures their shared local properties. In the topological version of Galois Theory functions of one variable it is proved that the character of location of the Riemann surface of a function over the complex line can prevent the representability of this function by quadratures.

ABSTRACT : In this paper it is shown that analytic continuation and germ of many-valued analytic function that set at least the topology of this set. This is needed to construct topological version for germ function of meromorphic function.

KEYWORDS : Analytic ,Open ,Monodromy ,Dimension,Meromorphic , Differential space

DEFINITION:

Suppose ƒ is an analytic function defined on a non empty open subset U of the complex plane C. If V is a larger open subset of C, containing U, and F is an analytic function defined on V such that

F (z) = ƒ(z) $\forall z \in U$

Then F is called an analytic continuation of ƒ

DEFINITION:

Two pairs $(f_1, \emptyset_1)$ and $(f_2, \emptyset_2)$ shall be equivalent if and only if $\emptyset_1 = \emptyset_2$ and$f_1 = f_2$ in some neighborhood of $\emptyset_1$.The conditions for an equivalence relation are obviously fulfilled. The equivalence classes are called germs, or more specifically germs of analytic function

The set of all germs F_φ with ϕϵD is called a sheaf over D; we shall denote it by σ or σ_D.If we are dealing with germs of analytic function,σ_D Is called the sheaf of germs of analytic function over D.

DEFINITION:

There is a projection map $\pi: \sigma \to D$ which maps F_φ on ϕ. For a fixed $\sigma \epsilon D$ the inverse image $\pi^{-1}(\varphi)$ is called the stalk over φ; it is denoted by σ_φ.

I.ON THE COUNTABILITY OF MULTIVALUED ANALYTIC FUNCTIONS

LEMMA: 1.1

Let a neighborhood U of the origin in the space C^n be the direct product $U = U_1 \times U_2$ of a connected neighborhood U_1 in the space C^{n-1} and a connected neighborhood U_2in the complex lineC^1. Then any function f that is analytic in the complement of the hyper plane $z = 0$ in the neighborhood U and is bounded of the origin can be continued analytically to the entire neighborhood$U$.

PROOF:

The lemma follows from the Cauchy integral formula. Indeed,

let us define a function $\bar{f}$ on the domain U by the Cauchy integral

$$\bar{f}(x,z) = \frac{1}{2\pi i}\int_{\gamma(x,z)} \frac{f(x,u)du}{u-z}$$

Where,

$x$and$z$ are points in the domains U_1 and $U_{2,}$ respectively.

$f(x,u)$is the given function, and $\gamma(x,z)$ is an integrating contour that belongs to complex line$\{x\} \times C^1$ in the domain U, en closes the points (x,z) and $(x,0)$, and continuously depends on (x,z).

The function $\bar{f}(x,z)$ defines the desired analytic continuation.

Indeed,

The function $\bar{f}$ is analytic in the entire domain U.

According to the Riemann theorem on a removable singularity,

This function coincides with the given function f in a neighborhood of the origin.

Hence the proof

THEOREM:1.2

If (f,D) is a function element and if γ is a curve which starts at the center of D,then (f,D)admits at most one analytic continuation along γ.

PROOF:

If γ is covered by chains $\zeta_1 = \{A_0,A_{1,}A_{2,} \dots, A_m\}$ and $\zeta_2 = \{B_0, B_1, B_2, \dots, B_n\}$, Where,

$$A_0 = B_0 = D.$$

If(f, D) can be analytically continued along ζ_1 to a function element (g_m, A_m),and

If (f, D)can be analytically continued along ζ_2 to (h_n, B_n),

Then,

$g_m = h_n$in$A_m \cap B_n$.

Since,

$A_m$andB_n are, By assumption, discs with the same centre $\gamma(1)$,

It follows that ,

$g_m$andh_n have the same expansion in powers of $z - \gamma(1)$, and we may as well replace A_m andB_n by whichever is the larger one of the two.

With this agreement, the conclusion is that $g_m = h_n$.

Let ζ_1 and ζ_2 be as above.

There are numbers ,

0= $s_0 < s_1 < \cdots < s_m = 1 = s_{m+1}$ and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = 1 = \sigma_{n+1}$

Such that ,

$\gamma([s_i, s_{i+1}]) \subset A_i, \gamma([\sigma_j, \sigma_{j+1}]) \subset B_j$,(0$\leq i \leq m, 0 \leq j \leq n$).

There are function elements,

$(g_i, A_i) \sim (g_{i+1}, A_{i+1})and(h_j, B_j) \sim (h_{j+1}, B_{j+1})$

For $0 \leq i \leq m - 1$ and $0 \leq j \leq n - 1$.

Here,

$$g_0 = h_0 = f.$$

We claim that,

if$0 \leq i \leq m$ and $0 \leq j \leq n$, and if $[s_i, s_{i+1}]$ intersects$[\sigma_i, \sigma_{i+1}]$.

Then,

$$(g_i, A_i) \sim (h_j, B_j).$$

Assume there are pairs (i, j) for which this is wrong.

Among them there is one for which $i + j$ is minimal.

It clear that than $i + j > 0$.

Suppose,

$s_i \geq \sigma_j$.then $i \geq 1$, and since $[s_i, s_{i+1}]$ intersects $[\sigma_j, \sigma_{j+1}]$.

We see that,

$$\gamma(s_i) \in A_{i-1} \cap A_i \cap B_j.$$

The minimality of$i + j$ shows that $(g_{i-1}, A_{i-1}) \sim (h_j, B_j)$;and

Since,

$(g_{i-1}, A_{i-1}) \sim (g_i, A_i)$,Implies that $(g_i, A_i) \sim (h_j, B_j)$.

This is contradicts our assumption.

The possibility $s_i \leq \sigma_j$.

In particular,

It holds for the pair(m, n).

Hence the proof.

THEOREM:1.4

Suppose $\{\gamma_t\} 0 \leq t \leq 1)$ is one- parameter family of curves from α and βin the plane, D is a disc with center at α, and the function element (f, D) admits analytic continuation along each γ_t, to an element (g_t, D_t). Then$g_1 = g_{0.}$

PROOF:

Fix $t \in I$. there is chain $\zeta = \{A_0, A_1 \dots A_n\}$ which covers γ_t, with$A_0 = D$,

Such that ,

$(g_t, D_t)$is obtained by continuation of (f, D) along ζ.

There are numbers $0 = s_0 < s_1 < \cdots < s_n = 1$

such that,

$$E_i = \gamma_t([s_i, s_{i+1}]) \subset A_i (i = 0,1,2, \ldots., n-1).$$

There exists an $\epsilon > 0$ which is less than the distance from any of the compact sets E_i to the corresponding open disc A_i.

The uniform continuity of φ on I^2 shows that there exists a $\delta > 0$.

Such that ,

$|\gamma_t(s) - \gamma_u(s)| < \in$ if $s \epsilon I, u \epsilon I, |u - t| < \delta$.

Suppose,

$u$satisfies these conditions

Then,

shows that ζ covers γ_u, and shows that both g_t and g_u are obtained by continuation of (f, D) along this same chain ζ.

Hence,

$$g_t = g_u.$$

Thus ,

Each $t \in I$ is covered by a segment J_t.

Such that,

$g_u = g_t$for all $u \epsilon I \cap J_t$.

Since,

$I$is compact, I is covered by finitely many J_t; and since I is connected , we see in a finite number of steps that $g_1 = g_0$.

Hence the proof

THEOREM:1.5

Suppose Ω is a simply connected region, (f, D) is a function element,$D \subset \Omega$ and (f, D) can be analytically continued along every curve in Ω that starts at the center of D. Then there exists $g \in H(\Omega)$ such that $g(z) = f(z)$ for all$z \in D$.

PROOF:

Let Γ_0 andΓ_1 be two curves in Ω from the center α of D to same point$\beta \in \Omega$.

It follows that ,

The analytic continuation of (f, D) along Γ_0 and Γ_1 lead to the same element (g_β, D_β),

Where,

$D_\beta$is a disc with center at β. If $D_{\beta_{1,}}$

Then,

$(g_{\beta_1}D_{\beta_1})$can be obtained by first continuing (f, D) to β

Then,

Along the straight line from β to $\beta_{1.}$

This shows that,

$g_{\beta_1} = g_\beta$in $D_{\beta_1} \cap D_{\beta.}$

Hence ,

$g(z) = g_\beta(z)$, $(z \in D_\beta)$ is there fore consistent and gives the holomorphic extension of f.

Hence the proof.

II.MODIFICATION OF TOPOLOGY OF AN ANALYTIC SET

LEMMA:2.1

Let a subset T of an $(n-1)$-dimensional analytic set $\sum$ belonging to an n-dimensional analytic manifold M have the following properties.

1. The set T is a real topological submaifold of$M$ of co-dimension two, i.e., any point $a\epsilon T$ has a neighborhood U_a in M such that the set $U_a \cap T$ is a topological sub manifold in the domain U_a of real dimensional $2n-2$.

2. The set $\sum\backslash T$ is a closd subset of $\sum$ of real co-dimension ≥ 2 $(i, e., \sum\backslash T$ is a union of finitely many real topological sub manifolds of M of dimension $\leq 2n-4)$.

Then any $(n-1)$-dimensional irreducible component of $\sum$ intersects exactly one connected component of the topological manifold T. More over, any connected component of T is dense in the corresponding irreducible $(n-1)$-dimensional component of the analytic set$\sum$.

PROOF:

Lemma is a consequence of the following facts:

a) A set of co-dimension two cannot separate a topological manifold,

b) If all singular points are deleted from a irreducible component of an analytic set, then the remaining manifold is connected.

c) Let us first show that

Any connected component T^0 of the set T intersects exactly one irreducible component of theset $\sum$.

Indeed, the set $\sum\backslash\sum_H$ is of real dimension $\leq 2n-4$;

Therefore,

This set cannot separate the connected$(2n-2)$-dimensional real manifold T^0into parts.

Thus,

The $D_i \cap \sum_{H.}$Since the set $D_i\backslash\sum_H$ is dense in the component D_i and the set D_i is closed,

It follows that,

$T^0$is entirely contained in the irreducible component D_i of the set $\sum$.

Suppose that,

A point a of T^0belongs to another $(n-1)$-dimensional component D_j,

$D_j \neq D_i, of \sum$.

However,

By assumption, the set T and hence its component T^0 are open in the topology of $\sum$.

Since,

The set $D_j \cap \sum_H$ is dense in D_j.

It follows that.

T^0contains some points of the set $D_j \cap \sum_{H,}$

Which is impossible.

Which is contradiction

This proves the desired assertion.

Let us now show that different connected components of the manifold T cannot belong to the same $(n-1)$-dimensional irreducible component of the set $\sum$.

Indeed,

If all singular points and all points not belonging to the manifold T are delete from an irreducible $(n-1)$-dimensional component , then a connected manifold is obtained.

Hence,

It is covered by exactly one connected component of the manifold$T$

This proves the lemma

III.TOPOLOGY OF GERMS OF MEROMORPHIC FUNCTIONS

THEOREM:3.1

In this case,0 is a typical value for the meromorpic germ $f=\frac{P}{Q}$if and only if the strict transfom of the curve $\{P=0\}$ intersects only components of the exceptional divisor $\mathfrak{D}$ with $K(E)\leq l(E)$.

PROOF:

Suppose that,

The value 0 is typical if and only if the family $P+cQ\ is\ \mu$-constant (for c from a neighborhood of 0)

If a family $P_c$of function of two variables $(c\in(\mathcal{C},0))$ is μ-constant then the embedded resolution of the curves $\{P_c=0\}$ are combinatorially equivalent.

However these resolution are obtained by blow-ups of different points and thus (minimal) resolution of the curve$\{P_0=0\}$can be not a resolution of the curve $\{P_c=0\}$.

Let us suppose that,

The strict transform of a branch of the curve$\{P=0\}$ intersects a component E of the exceptional divisor with $K(E)>l(E)$. In this in local coordinates at the point of intersection $\widetilde{p}$ $=u^k.y\ and\ Q=v.x^l$ $(u(0)\neq 0,v(0)\neq 0)$.

The lifting $\widetilde{P}$ +$c\ \widetilde{Q}$ of the function $P+cQ$ is equal to$x^l(u.x^{k-l}.y+c.v)$.

Therefore,

Its multiplicity along the component E is equal to l and it is less than that one of the function p. Thusπ is not the minimal resolution of $\{P+cQ=0\}$ for $c\neq 0$. this is contradiction.

If the strict transform of the curve$\{P=0\}$ intersect only components of the exceptional divisor with $K(E)\leq l(E)$,

Then the family $\widetilde{p}$ +$c\widetilde{Q}$ in a neighborhood of the intersection has the form $x^k(u.y+cv.x^{l-k})$ with $l-k\geq 0$.

Thus the strict transform of the corresponding branch of the exceptional divisor.

Therefore $\{P + cQ = 0\}$ has the same resolution as $\{p = 0\}$ and the family is μ-constant.

THEOREM:3.2

Let $f = \frac{P}{Q}$ be a germ of meromorphic function of two variables . Then.

a) If the germ of the curve $\{P = 0\}$ at 0 has a non-isolated singularity but $\{P + cQ = 0\}$ has an isolated singularity (for c small enough) then the value 0 is atypical .

b) If $P = R.P_1$ and $Q = R.Q_1$ where $R = g.c.d.(P, Q)$ and the curve $\{P_1 = 0\}$ has an isolated singularity at the origin then 0 is a typical value for meromorphic germ f if and only if $(\mathcal{M}_f^0) = 0$.

PROOF:

The first part follows from the definition of typical value.

Let us assume that,

$\{P = 0\}$has an isolated singularity at the origin.

If $Q_1(0) \neq 0$

Then,

$\boldsymbol{\mathcal{X}}(\mathcal{M}_f^0) = \mathcal{X}(\{x, y) \in B_\varepsilon : P_1 = c\} \setminus \{R = 0\}) = 1 - \mu(P_1, 0\} - (P_1, R)_{0,}$

Where,

$(P_1, R)_0$ is the intersection multiplicity of the both curves at the origin.

Therefore,

The eular characteristic $(\mathcal{M}_f^0)$ is equal to zero if and only if P_1 has no critical point at the origin and $(P_1, R)_0 = 1$.

It means that we are in the case

$$f = \frac{P}{Q} = \frac{xy}{x}$$

If $Q_1(0) = 0$

Then, it follows from that the Euler character

$\mathcal{X}(\mathcal{M}_f^0)$= $-\mu(P, 0)$ +$\sum_{A \in \{P+cQ=0\} \cap B_\varepsilon} \mu(P + cQ, A)$.

Let k(respectively s) be the intersection multiplicity at the origin of the curve $\{R = 0\}$ with thecurve $\{P_1 = 0\}$ (respectively with the curve $\{P_1 + cQ_1 = 0\}$).

At any other intersection point $A \in \{P_1 + cQ_1 = 0\} \cap B_\varepsilon\} \setminus \{0\}$ the curve $\{P + cQ = 0\}$ has a non-degenerate critical point with Milnor number equals to 1.

Let l be the number of such points.

The conservation law of the intersection multiplicity gives

$$k = (R, P_1)_0 = (R, P_1 + cQ_1)_0 + l = s + l.$$

Using the following formula for the Milnor number

$$\mu(RP_1, 0) = (\mu(P_1, 0) + 2(R, P_1)_0 - 1$$

And the vanishing of the Euler characteristic $\mathcal{X}(\mathcal{M}_f^0)$ on has

0= $\mathcal{X}(\mathcal{M}_f^0)$= $\big(\mu(P_1 + \mathrm{c}Q_1, 0) - \mu(\mathrm{P}_1, 0)\big) + (\mathrm{R}, \mathrm{P}_1 + \mathrm{c}Q_1)_0 - (\mathrm{R}, \mathrm{P}_1)_0$.

Since,

The Milnor number and the intersection multiplicity are semi continuous,

The family $P_1 + cQ_1$ has to be μ-constant and $(R, P_1 + cQ_1)_0 = (R, P_1)_0$.

Note that these two last conditions are equivalents to the fact that the family $P + cQ$ is μ-constant.

Now the proof that 0 is typical follows from the proof of the "only if" part in the general case follows from the fact if

$R = g.c.d(P, Q) = R_1^{n-1} \ldots . R_S^{n_s}andQ = R.Q_1$ and $Q = R.Q_1$

Then,

The meromorphic germ f defines the same fibration as the meromorphic germ $f = \frac{R_1 \ldots . R_S . P_1}{R_1 \ldots R_S . Q_1}$.

REFERENCES:

[1] A.G. Khovanskii, *"Therepresentability of functions by quadratures,"*Usp. Mat. Nauk,26,No.4,251-252 **(1971).**

[2] A.G. Khovanskii, "*Riemannian surfaces of functions that are representable byquadratures ,"* in abstracts of the VI All-Union Topological conference, Tbilisi, **(1972),**p.125.

[3] A.G. Khovanskii, *On the representability of functions by quadratures, Candidatethesis,* V.A. Steklov Mathematics Institute, Russian Academy of Sciences,**(1973).**

[4] A.G. Khovanskii, *"Topological obstructions to the representability of functions byquadratures,"*J.Dynam. Control Systems,1, No. 1, 91-123 **(1995)**

[5] B.A. Fuks, *Introduction to the Theory of Analytic Functions of Several ComplexVariables [in Russian],*Fizmatlit, Moscow,**(1962).**

[6] M.Goreskyand R.Macpherson, *stratified Morse Theory,* Springer-Verlag002York**,(1988).**

[7] A'CAMPO N. La function zeta *d'unemonodromy,* Comment. Math. Helv.50**(1975),** 233-248

[8] ARNOLD V,I., GUSEIN-ZADE S.M., VARCHENKO A.N., *Singularities ofDifferentiable maps,* Vol.II , Birkhauser, Boston-Basel-Berlin.**(1988).**