Spaces of Distribution for Fourier-Stieltjes Transform of Vector Measures on Compact Groups

B.G.Khedkar[#1], S. B. Gaikwad[*2]
[#]Arts, Commerce and Science College Sonai, Tal.:Newasa,
Dist.: Ahmednagar, Pin: 414105, (M.S) India
[*]New Arts, Commerce and Science College Ahmednagar,
Pin: 414005, (M.S) India.

Abstract:

This paper deals with distributional spaces of Fourier-Stieltjes transform of vector measures on compact groups.The present paper mainly provides some topological properties of functional spaces.In particular we found dual spaces.

AMS Mathematics Subject Classification: (2010) 46 F12

Keywords: *Fourier transform, Stieltjes transform, Compact group, Distribution, Banach Space.*

1. INTRODUCTION

The Fourier transform of a complex valued function on a commutative locally compact group G, such as $\mathbb{R}^n$, is again a complex valued function on the character group X of G. Otherwise, it is family (E_σ) $\sigma \in \Sigma$ of continuous linear operators $E_\sigma: H_\sigma \to H_\sigma$, where$\Sigma$ is the dual object of the compact non commutative group G, and σ a class of irreducible unitary representations of G in Hilbert space H_σ.

In case $\mathbb{C}$ is replaced by a Banach space FS_α, it is a family of continuous sesquilinear mappings $\phi(\sigma): H_\sigma \times H_\sigma \to FS_\alpha$. In fact, for each$\sigma \in \Sigma$, we choose once and for all an element U^σ in σ, denote its representation space by H_σ , and fix an orthonormal basis $(\xi_1^\sigma, \dots\dots.\xi_{d_\sigma}^\sigma)$ of H_σ , where $d_\sigma = \dim H_\sigma$, as a canonical basis. We put $u_{ij}^\sigma(t) = \langle U_t^\sigma \xi_j^\sigma, \xi_i^\sigma\rangle$ and introduce the operator $\overline{U^\sigma}$ on H_σ such that $\langle \overline{U}_t^\sigma \xi_j^\sigma, \xi_i^\sigma\rangle = \overline{u_{ij}^\sigma}(t)$, the complex conjugate of $u_{ij}^\sigma(t)$. The Fourier- Stieltjes transform on G for an FS_α-valued bounded vector measure m, where FS_α is a normed space is given by ;

$$\hat{m}(\sigma)(\xi,\eta) = \int_G \langle \overline{U}_t^{\,\sigma} \xi, \eta\rangle dm(t) \ (\xi,\eta) \in H_\sigma \times H_\sigma .$$

(For details on vector measures see [5] and [6]). The mapping $H_\sigma \times H_\sigma \to FS_\alpha$, $(\xi, \eta) \to \hat{m}(\sigma)(\xi,\eta)$ is a continuous and sesquilinear [2]This generates a certain number of interesting spaces $S_p(\Sigma, FS_\alpha)$ that we specify as follows.

We write $\prod_{\sigma\in\Sigma} S(H_\sigma X H_\sigma, FS_\alpha) = S(\Sigma, FS_\alpha)$ is space of continuous sequilinear mapping from $H_\sigma \times H_\sigma$ into FS_α. $S(\Sigma, FS_\alpha)$ is a linear space with addition and multiplication by scalars, defined coordinate wise. For $\in S(\Sigma, FS_\alpha)$, we put:

$$\|\phi\|_\infty = \sup \{\|\phi(\sigma)\| / \sigma \in \Sigma\}$$

With $\|\phi(\sigma)\| = \sup\{\|\phi(\sigma)(\xi,\eta)\| / \|\xi\| \le 1, \|\eta\| \le 1\}$.we denote by $S_\infty(\Sigma, FS_\alpha)$, the space $\{\phi \in S(\Sigma, FS_\alpha) / \|\phi\|_\infty < \infty\}$

$S_{00}(\Sigma, FS_\alpha)$, the space $\{\phi \in S_\infty(\Sigma, FS_\alpha) \,|\, \{\sigma \in \Sigma \,|\, \phi(\sigma) \ne 0\} is finite\}$ and $S_0(\Sigma, FS_\alpha)$ is the space $\{\phi \in S_\infty(\Sigma, FS_\alpha) \forall \varepsilon > 0, \{\sigma \in \Sigma \,|\, \|\phi(\sigma)\| > \varepsilon\} is finite\}$

In[3] the author proved that:

(1) The mapping $\phi \to \|\phi\|_\infty$ is a norm on $S_\infty(\Sigma, A)$, and $S_\infty(\Sigma, A)$ is a banach space with respect to this norm.
(2) $S_{00}(\Sigma, A)$ is dense in $S_0(\Sigma, A)$.
(3) Every $\phi(\sigma) \in S(H_\sigma \times H_\sigma, A)$ is determined by the d_σ^2 elements $a_{ij}^\sigma = \phi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)$ of A. More precisely, we have: $\phi(\sigma) = \sum_{i,j=1}^{d_\sigma} d_\sigma a_{ij}^\sigma \hat{u}_{ij}^\sigma(\sigma)$, $\hat{u}_{ij}^\sigma$ being Fourier transform of u_{ij}^σ

2. DEFINITIONS

2.1. Test Function Space: The Space FS_α

A function f defined on $0 < t < \infty$, $0 < x < \infty$ is said to be member of FS_α if ϕ (t, x) is smooth for each non-negative integer l, q.

$$\gamma_{k,p,l,q}\phi(t,x) = \sup_I \left| t^k (1+x)^p D_t^l (xD_x)^q \phi(t,x) \right| \qquad (2.1)$$

$$\leq C_{lq} A^p . p^p \qquad p = 1, 2, 3, \text{------------}$$

Where the constant A and C_{lq} depend on the testing function ϕ.

The space FSα areequiparallel with their natural Housdoff locally topology τ_α. This topology is respectively generated by the total families of semi norms $\{\gamma_{k,p,l,q}\}$ given by (2.1).

2.2. Distributional Fourier-Stieltjes transform of generalized function in FS_α^*

Let FS_α^* is the dual space FS_α. This space FS_α^* consists of continuous linear function on FS_α.

Let $\phi(t,x) \in FS_\alpha^*$, for some s >0 and k > Re p, then distributional Fourier-Stieltjes Transform F(s, y) of FS {f (t, x)} = F(s, y) = $\langle f(t,x), e^{-ist}(x+y)^{-p} \rangle$ (2.2)

Where for each fixed t ($0 < t < \infty$), x ($0 < x < \infty$) the right side of above equation has same as an application of $f(t,x) \in FS_\alpha^*$ to $e^{-ist}(x+y)^{-p} \in FS_\alpha$.

3. MAIN RESULTS:

3.1 The Space $S_p(\Sigma, FS_\alpha)$ $1 \leq p \leq \infty$

We define:

$$S_p(\Sigma, FS_\alpha) = \{\phi \in S(\Sigma, FS_\alpha) \mid \sum_\sigma d_\sigma \sum_{ij} \left\| \phi(\sigma)(\xi_j^\sigma, \xi_i^\sigma) \right\|^p < \infty\}, \quad 1 \leq p < \infty,$$

and $S_\infty(\Sigma, FS_\alpha)$ as in the introduction. They are linear spaces for point wise operations.

We define a norm on $S_p(\Sigma, FS_\alpha)$ by

$$\|\phi\|_p = \left(\sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j} \left\| \phi(\sigma)(\xi_j^\sigma, \xi_i^\sigma) \right\|\right)^{1/p}$$

Theorem 3.1: For each p, $1 \leq p \leq \infty$, the space $S_p(\Sigma, FS_\alpha)$ is a Banach space.

Proof. The space p = ∞ was in done in [2]

Let (ϕ_n) be a Cauchy sequence from the space **S_p(**Σ, FS_α). Then $\sigma \in \Sigma$, the sequence ($\phi_n(\sigma)$)$_n$ is a Cauchy sequence from the space S($H_\sigma \times H_\sigma$, FS_α) which is known to be a Banach space. Thus there exists $\phi(\sigma) \in S(H_\sigma \times H_\sigma, FS_\alpha)$ such that

$$\lim_{n\to\infty}\|\phi(\sigma_n)-\phi(\sigma)\|=0 \quad (1)$$

Set $\alpha_{ij}=\phi(\sigma)(\xi_j,\xi_i)$ and for all n, $a_{ij}^n=\phi_n(\sigma)(\xi_j,\xi_i)$.

We consider ε > 0. Since (ϕ_n) is a Cauchy sequence , then there exist $n_0 \in N$ such that

$$\forall r,s\geq n_0, \|\phi_r-\phi_s\|_p<\varepsilon^{1/p} \quad (2)$$

$$\text{i.e. } \sum_\sigma d_\sigma \sum_{i,j}\left\|a_{ij}^{\sigma,r}-a_{ij}^{\sigma,s}\right\|^p<\varepsilon \ (3)$$

Letting s tends to infinity in (3), we have

$$\sum_\sigma d_\sigma \sum_{i,j}\left\|a_{ij}^{\sigma,r}-a_{ij}^{\sigma}\right\|^p<\varepsilon \ (4)$$

$$\text{i. e. } \|\phi_r-\phi\|_p<\varepsilon \quad \text{Pour } r\leq n_0 \ (5)$$

We have $\|\phi\|_p = \|\phi-\phi_r+\phi_r\|_p$

$\leq \|\phi-\phi_r\|_p+\|\phi_r\|_p$

$\leq \varepsilon + \|\phi_r\|_p < \infty$

Hence $\phi \in S_p(\Sigma, FS_\alpha)$. Finally (5) shows that (ϕ_n) converges to ϕ in $S_p(\Sigma, FS_\alpha)$.□

3.2 Duality in spaces $S_p(\Sigma, FS_\alpha)$

Theorem 3.2. Let p, q be such $1 \leq p \leq \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and FS_α^* be the dual of FS_α .Then the space (S_p (Σ, FS_α))* is isometric to S_q (Σ, FS_α^*).

Proof: The proof of the case p =1 (which implies q = ∞) can found in [9]. Now let 1 < p <∞. Let T:Sq (Σ, FS_α^*) →(Σ, FS_α))*, $\varphi \mapsto T_\varphi$ be defined by $\langle T_\varphi,\psi\rangle=\sum_{\sigma\in\Sigma}d_\sigma\sum_{i,j}\langle b_{ij}^\sigma,a_{ij}^\sigma\rangle, \psi\in S_p(\Sigma,FS_\alpha)$

Where $b_{ij}^\sigma=\varphi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)$ and $a_{ij}=\psi(\sigma)(\xi_j^\sigma,\xi_i^\sigma)$. Then theorem is the consequence of the following three lemmas.

Lemma 3.3 The mapping is linear and bounded.

Proof. The linearity of T is trivial. Let us show that it is bounded.

$$\text{We have } |\langle T_\varphi,\psi\rangle|=\left|\sum_{\sigma\in\Sigma}d_\sigma\sum_{i,j}\langle b_{ij}^\sigma,a_{ij}^\sigma\rangle\right|$$

$$\le \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j} \left|\left\langle b_{ij}^\sigma, a_{ij}^\sigma \right\rangle\right|$$

$$\le \sum_{\sigma\in\Sigma} d_\sigma \sum_{i,j} \left\|b_{ij}^\sigma\right\| \left\|a_{ij}^\sigma\right\|$$

$$\le \sum_{\sigma\in\Sigma}\sum_{i,j} d_\sigma^{1/q} \left\|b_{ij}^\sigma\right\| d_\sigma^{1/p} \left\|a_{ij}^\sigma\right\|$$

$$\le \left(\sum_{\sigma\in\Sigma}\sum_{i,j} d_\sigma \left\|b_{ij}^\sigma\right\|^q\right)^{1/q} \left(\sum_{\sigma\in\Sigma}\sum_{i,j} d_\sigma \left\|a_{ij}^\sigma\right\|^p\right)^{1/p}$$

$$\le \|\varphi\|_q \|\psi\|_p$$

So that $\|T_\varphi\| \le \|\varphi\|_q$ and therefore T is bounded with $\|T\| \le 1$.□

Lemma 3.4 The equality $\|T\|= 1$ holds.

Proof. From part 1; we have $\|T\| \le 1$. Let us show that $\|T\| \ge 1$.

Take a $\in FS_\alpha$, Such that $\|a\| = 1$. Since a $\neq 0$, we know from Functional analysis that there exists $b^* \in FS_\alpha^*$ such that $\|b^*\| = 1$ and $\langle b^*, a\rangle = \|a\| = 1$.

Given $\tau \in \Sigma$ we use the Kronecker symbol δ_{ij} to define $\psi_\tau \in S_p(\Sigma, FS_\alpha)$ by

$$\psi_\tau(\sigma)(\xi_j^\sigma, \xi_i^\sigma) = a_{ij}^\sigma = \begin{cases} d_\tau^{\frac{-2}{p}} a\delta ij & \text{if } \sigma = \tau \\ 0 \, if \, \sigma \neq \tau \end{cases}$$

and ϕ_τ in $S_q(\Sigma, FS_\alpha^*)$ by :

$$\varphi_\tau(\sigma)(\xi_j^\sigma, \xi_i^\sigma) = b_{ij}^\sigma = \begin{cases} d_\tau^{\frac{-2}{q}} b^* \delta ij & \text{if } \sigma = \tau \\ 0 \, if \, \sigma \neq \tau \end{cases}$$

We have $\|\varphi\|_q^q = \sum_\sigma d_\sigma \sum_{ij} \left\|b_{ij}^\sigma\right\|^q = \sum_\sigma d_\sigma \sum_{ij} \left\| d_\tau^{\frac{-2}{q}} b^* \delta_{ij} \right\|^q = d_\tau d_\tau d_\tau^{-2} = 1$

And $\|\psi\|_p^p = \sum_\sigma d_\sigma \sum_{ij} \left\|a_{ij}^\sigma\right\|^p = \sum_\sigma d_\sigma \sum_{ij} \left\| d_\tau^{\frac{-2}{p}} a \delta_{ij} \right\|^p = 1$

As such, $\left\langle T_\varphi, \psi_\tau \right\rangle = \sum_\sigma d_\sigma \sum_{ij} \left\langle b_{ij}^\sigma, a_{ij}^\sigma \right\rangle$

$$= \sum_\sigma d_\sigma \sum_{ij} \left\langle d_\tau^{\frac{-2}{q}} b^* \delta_{ij}, d_\tau^{\frac{-2}{p}} a \delta_{ij} \right\rangle$$

$$= d_\tau \sum_i \left\langle d_\tau^{\frac{-2}{q}} b^*, d_\tau^{\frac{-2}{p}} a \right\rangle$$

$$= d_\tau^2 \left(d_\tau^{\frac{1}{p}+\frac{1}{q}} \right)^{-2} \left\langle b^*, a \right\rangle = 1 = \|\varphi\|_q \|\psi\|_p$$

Hence $\|T\| \geq 1$. Finally $\|T\| = 1$.

Lemma 3.5. The mapping T is surjective.

Proof: In fact $f \in (S_p(\Sigma, FS_\alpha))^*$. for $\tau \in \Sigma$, let

$V_\tau = \{\psi \in S_p(\Sigma, FS_\alpha) \mid \psi(\sigma) = 0 if \sigma \neq \tau, \sigma \in \Sigma\}$

For $\psi \in V_\tau$, let $a_{ij}^\tau = \psi(\tau)(\xi_j^\tau, \xi_i^\tau), i, j = 1,2,.......d_\tau$. There exist linear forms $b_{ij} \in FS_\alpha^*, i, j = 1,2,.........., d_\tau$ such that $\langle f, \psi \rangle = d_\tau \sum_{ij} \langle b_{ij}^\tau, a_{ij}^\tau \rangle$. In fact, given d_τ^2 scalars λ_{ij}^τ such that $\sum_{i,j} \lambda_{ij}^\tau = \frac{\langle f, \psi \rangle}{d_\tau}$, there exists $b_{ij} \in FS_\alpha^*$ with $\langle b_{ij}, a_{ij}^\tau \rangle = 1$; denoting $b_{ij}^\tau = \lambda_{ij}^\tau b_{ij}$, we have what is required.

Now let us consider an element ϕ of $S_{00}(\Sigma, FS_\alpha)$

Since $S_{00}(\Sigma, FS_\alpha)$ is subset of $S_2(\Sigma, FS_\alpha)$, one can write according to Riesz-Fischer theorem,

$\phi = \sum_{\tau \in \Sigma} d_\tau \sum_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau$. In fact, there exists a finite subset Σ' of Σ such that $\phi = \sum_{\tau \in \Sigma'} d_\tau \sum_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau$

Putting $\phi_\tau = d_\tau \sum_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau$ we have $\phi = \sum_{\tau \in \Sigma'} \phi_\tau$. It is clear that ϕ_τ belongs to V_τ because for $\sigma \neq \tau$, $\hat{u}_{ij}^\sigma(\sigma) = 0$ (Schur'sorthogonally property), so $\phi_\tau = \sum_{ij} a_{ij}^\tau \hat{u}_{ij}^\tau(\sigma) = 0$ .Thus there exists linear forms $b_{ij} \in FS_\alpha^*, i, j = 1,2,........., d_\tau$ such that

$$\langle f, \phi_\tau \rangle = d_\tau \sum_{i,j} \langle b_{ij}^\tau, a_{ij}^\tau \rangle$$

Now by linearity of f

$$\langle f, \phi \rangle = \sum_{\tau \in \Sigma'} d_\tau \sum_{i,j} \langle b_{ij}^\tau, a_{ij}^\tau \rangle$$

Defining φ by:

$\varphi(\tau)\ (\xi_j^\tau, \xi_i^\tau) = b_{ij}^\tau$ if $\tau \in \Sigma'$ and $\varphi(\tau)\ (\xi_j^\tau, \xi_i^\tau) = 0$ otherwise , we have $\varphi \in S_{00}(\Sigma, FS_\alpha^*)$ and $\langle f, \phi \rangle = \langle T_\varphi, \phi \rangle$. This means that the continuous linear forms of f and T_φ coincide on $S_{00}(\Sigma, FS_\alpha)$ which is dense subset of $S_p(\Sigma, FS_\alpha)$.

Hence $f = T_\varphi$.

The three lemmas show that T is an isometry from $S_p(\Sigma, FS_\alpha^*)$ onto $(S_p(\Sigma, FS_\alpha))^*$.

REFERENCES

[1] AssiamouaV.S.K., 1989, L_1 (G, A)-multipliers, Acta Sci. Math., 53, pp.309-318.

[2] AssiamouaV.S.K., OlubummoA., 1989, Fourier-Stieltjes Transform of Vector Measures on Compact Groups, Acta Sci. Math.,53,pp.301-307.

[3] Assiamoua V.S.K.,Mansah Y., The Fourier Algebra A_1 (G,A) of vector -valued functions on compact groups, Contemporary Problems in Mathematical Physics,pp.223-230.

[4]Diestel J.,Uhl Jr. J. J., 1977, Vector Measures , Amer. Math-Soc, Math surveys,n15.

[5]Effros E., 1991, A new approach to operator spaces, Canadian Math., Bull., 34, pp.329-337.

[6]Gaikwad S.B., 2015Generalized Fourier-Modified Stieltjes Transforms, Proceedings of Role of Mathematics in Science Engineering and Technology.

[7]Hewitt E.,Ross K.A., 1970, Abstract Harmonic Analysis, Volume I and II,Springer-Verlag, New-York-Berlin-Heidelberg.

[8]Mensah Y.,Assiamoula V,S.K., 2009, The dual of Fourier Algebra A_1(G,A) of vector- valued functions on compact groups,Afr. Diaspora J.Math.,Vol. 8, Number 1,pp.28-34.

[9]Mensah Y., V, S .K., Assiamoula, 2010,On Spaces of Fourier-Stieltjes Transform of Vector Measures on Compact Groups,Mathematical Sciences,Vol.4,No.1,pp.1-8,

[10]Pathak R.S., 2001, A Course in Distribution Theory and Applications, CRC Press.

[11] Pisier G., 1998, Non-commutative vector valued L_p-spaces and completely p-summing maps, Asterisque 247. SocMath.France,.

[12]Sharma V.D. and Dolas P.D., 2012, Analyticity of Distribution Generalized Fourier-Stieltjes Transforms, International Journal of Mathematical Analysis, Vol.6, No.9, pp.447-451,.