Solution of Fractional Ordinary Differential Equations by Rangaig Transform Method

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Abstract: In this paper a method is introduced to find the solution of fractional ordinary differential equation by using Rangaig integral transform. We then proceed to give some basic properties and derive the Rangaig transform of fractional order.

***Keywords**:* Rangaig transform, fractional derivative.

1. Introduction:

The solution for nonlinear fractional differential equation, and their computation has been studied for over fifty years now and has helped in finding solutions to problems in several theoretical and applied sciences such as, electrical control theory, hydro-dynamics, thermo-dynamics, signal processing, solid state physics, fibre optics, theoretical biology, ecology , viscoelasticity , and stochastic based finance , different authors have pointed out these and many other uses, for example works by Podlubny [9]. Our aim is to exhibit exact solution of some non-homogeneous fractional ordinary differential equation by using the Rangaig transform method . We apply the Rangaig transform method to obtain a new exact solution of some fractional ordinary differential equations.

***2. Fundamental properties of fractional calculus*:**

The association of differential integral transform with fractional integers and derivatives are used to solve different types of differential and integral equations. A derivative of fractional order, in Abel-Riemann sense [2013] (A-R) is given by

$$D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha \le m \\ \frac{d^m}{dt^m} f(t) & ,\alpha = m \end{cases} \quad \ldots (1)$$

where $m \in Z^+, \alpha \in R^+$ and D^{α} is the derivative operator defined by Abel – Riemann, and

$$D^{-\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad ,0 < \alpha \le 1 \quad \ldots (2)$$

An integral of fractional order is also, according to Abel – Riemann, defined by implementing the integration operator J^{α} as given below:

$$J^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad ,t > 0, \alpha > 0 \quad \ldots (3)$$

Podlubny's work [9] on the fundamental properties of fractional integration and fractional differentiation are hereby given respectively as

$$J^{\alpha}[t^n] = \frac{\Gamma[1+n]}{\Gamma[1+n-\alpha]} t^{n+\alpha} \quad \ldots (4)$$

$$D^{\alpha}[t^n] = \frac{\Gamma[1+n]}{\Gamma[1-n+\alpha]} t^{n-\alpha} \quad \ldots (5)$$

We also consider the definition of fractional derivatives which is given by

$${}^{c}D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma[m-\alpha]} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha \le m \\ \frac{d^m}{dt^m} f(t) & ,\alpha = m \end{cases} \quad \ldots (6)$$

Where $\frac{d^m f(t)}{dt^m}$ is the m[th] order derivative of $f(t)$ with respect to t.

A fundamental property of the Caputo fractional derivative is given by the following [9]

$$J^{\alpha}\left[{}^{c}D^{\alpha}[f(t)]\right] = f(t) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{t^k}{k!} \quad \ldots (7)$$

3. Basic features of Rangaig Transform[1]:

The Rangaig Transform is defined over the set of function

$$\mathrm{H} = \begin{Bmatrix} \mathrm{h(t)}: \exists N, \lambda_1, \lambda_2 > 0, |h(t)| > Ne^{\lambda_1 |t|} \\ t \in (-1)^{i-1} \times (-\infty, 0) \end{Bmatrix} \quad \ldots (8)$$

Where in the given function H (above), the constant N (which is arbitrary) must be finite and the constants λ_1, λ_2 can be finite or infinitely finite.

Then the Rangaig Transform denoted by the operator η is defined by the integral equation

$$\eta[h(t)] = \Lambda(\mu) = \left(\frac{1}{\mu}\right)\int_{-\infty}^{0} e^{\mu t} h(t)dt, \ where \frac{1}{\lambda_1} \leq \mu \leq \frac{1}{\lambda_2} \qquad \ldots (9)$$

Rangaig Transform of a Derivative:

Let the function h(t) $\in H$ and its n^{th} order derivative (with respect to t) exist in the set (8), then the Rangaig Transform of the n^{th} derivative of h(t) is given by

$$\eta\left[h^{(n)}(t)\right] = {}_n\Lambda(\mu) = (-1)^n\mu^n\Lambda(\mu) + (-1)^{n+1}\sum_{k=1}^{\infty}(-1)^n\mu^{n-2-k}h^{(k)}(0) \qquad \ldots (10)$$

By putting n = 1, 2, 3, … in equation (10) we get the Rangaig Transform of first, second, and so on derivatives of h(t) with respect to 't' as shown below:

$$\eta[h'(t)] = {}_1\Lambda(\mu) = \left(\frac{1}{\mu}\right)\int_{-\infty}^{0} e^{\mu t}h^{'(t)}dt = -\mu\Lambda(\mu) + \frac{1}{\mu}h(0) \qquad \ldots (11)$$

$$\eta[h''(t)(t)] = {}_2\Lambda(\mu) = \left(\frac{1}{\mu}\right)\int_{-\infty}^{0} e^{\mu t}h^{''(t)}dt = \mu^2\Lambda(\mu) + \frac{1}{\mu}h^{'}(0) - h(0) \qquad \ldots (12)$$

$$\eta[h'''(t)] = {}_3\Lambda(\mu) = \left(\frac{1}{\mu}\right)\int_{-\infty}^{0} e^{\mu t}h^{'''(t)}dt = -\mu^3\Lambda(\mu) + \frac{1}{\mu}h^{''}(0) - h'(0) + h(0) \qquad \ldots (13)$$

Convolution theorm:

Let $h(t)\ and\ r(t)$ be two functions and $h(t) \& r(t) \in H,$ then the Rangaig transform for the convolution identity [1] of the two functions is given by

$$(h * r)(t) = \int_0^t h(t-T)r(T)dT \qquad \ldots (14)$$

And is defined by

$$\eta[h(h * r)] = -\mu\Lambda_1(\mu)\Lambda_2(\mu) \qquad \ldots (15)$$

Where $\Lambda_1(\mu)$ and $\Lambda_2(\mu)$ are the Rangaig transforms for h(t) and r(t) respectively.

Proposition 1:

If $\Lambda(\mu)$ is the Rangaig transform of the function h(t) then the Rangaig transform of fractional derivative of order α is defined by

$$\eta\left[D^{-\alpha}[h(t)]\right] = {}_\alpha\Lambda(\mu) = (-1)^\alpha\mu^\alpha\Lambda(\mu) + (-1)^{\alpha+1}\sum_{k=1}^{n}(-1)^\alpha\mu^{\alpha-2-k}h^{(k)}(0)D^{\alpha-k}\, h(t)dt \ldots (16)$$

Proposition 2:

If $\Lambda(\mu)$ is the Rangaig transform of the function h(t) then the Rangaig transform of a general fractional ordinary linear differential equation of order α is given as

$$\frac{\partial^\alpha u(t)}{\partial t^\alpha} = \frac{\partial^2 u(t)}{\partial t^2} + \frac{\partial u(t)}{\partial t} + u(t) + c \qquad \ldots (17)$$

With initial condition $u(0) = f(0)$

Solution:

We proceed by taking the Rangaig transform of both sides of the equation (17),we obtain

$$\eta\left[\frac{\partial^\alpha u(t)}{\partial t^\alpha}\right] = \eta\left[\frac{\partial^2 u(t)}{\partial t^2}\right] + \eta\left[\frac{\partial u(t)}{\partial t}\right] + \eta[u(t)] + \eta[c]$$

$$\mu^\alpha[(-1)^\alpha\Lambda(\mu) - \mu^2\Lambda(\mu) + \mu\Lambda(\mu) - \Lambda(\mu)] = \frac{1}{\mu}h^{'}(0) - h(0) + \frac{1}{\mu}h(0) + \frac{c}{\mu^2}$$

$$\Lambda(\mu)[\mu^\alpha(-1)^\alpha - \mu^{\alpha+2} + \mu^{\alpha+1} - \mu^\alpha] = \frac{1}{\mu}h^{'}(0) - h(0) + \frac{1}{\mu}h(0) + \frac{c}{\mu^2}$$

$$\Lambda(\mu) = [\frac{1}{\mu}h^{'}(0) - h(0) + \frac{1}{\mu}h(0) + \frac{c}{\mu^2}]\frac{1}{\mu^\alpha(-1)^\alpha - \mu^{\alpha+2} + \mu^{\alpha+1} - \mu^\alpha}$$

we thus obtain the solution by taking the inverse Rangaig transform both sides as shown below

$$h(t) = \eta^{-1}\{[\frac{1}{\mu}h^{'}(0) - h(0) + \frac{1}{\mu}h(0) + \frac{c}{\mu^2}]\frac{1}{\mu^\alpha(-1)^\alpha - \mu^{\alpha+2} + \mu^{\alpha+1} - \mu^\alpha}\}$$

3. Applications of Rangaig transform to Non-homogeneous Fractional Ordinary Differential Equations.

In this part we are using the Rangaig Transform method to find the solution of non-homogeneous fractional ordinary differential equation as follows.

Example 1. Obtain the solution of the following fractional ordinary differential equation.

$D^{\frac{3}{2}}[y(t)] + Dy(t) = 1 + t$ …(18)

With initial condition y(0) = 0 = y'(0) & $[D^{-\frac{1}{2}}h(t)dt]_{t=0} = 0$

Solution:

In order to solve the above non-homogeneous fractional ordinary differential equation we proceed by taking the Rangaig transform of both sides

$\eta\left[D^{\frac{3}{2}}[y(t)]\right] + \eta[Dy(t)] = \eta[1] + \eta[t]$ …(19)

$$\mu^{\frac{3}{2}}\left[(-1)^{\frac{3}{2}}\Lambda(\mu) + (-1)^{\frac{3}{2}}\sum_{k=1}^{\frac{3}{2}}(-1)^k\mu^{\frac{3}{2}-(-2-k)}[D^{\frac{1}{2}-k}\,h(t)]_{t=0}\right] - \mu\Lambda(\mu) + \frac{1}{\mu}h(0) = \frac{1}{\mu^2} - \frac{1}{\mu^3}$$

Now taking k=1 and applying the initial conditions finally we get

$$-\mu\Lambda(\mu) = \frac{1}{\mu^2} - \frac{1}{\mu^3}$$

$$\Lambda(\mu) = \left[\frac{1}{\mu^2} - \frac{1}{\mu^3}\right]\left(-\frac{1}{\mu}\right)$$

$$\Lambda(\mu) = \frac{1}{\mu^4} - \frac{1}{\mu^3}$$

To get the required solution we now take the inverse Rangaig transforms both side

$$\eta^{-1}[\Lambda(\mu)] = \eta^{-1}\left[\frac{1}{\mu^4}\right] + \eta^{-1}\left[-\frac{1}{\mu^3}\right]$$

And we get $y(t) = \frac{1}{2}t^2 + t$

Example 2. Find the solution of the following non-homogeneous fractional ordinary differential equation

$D^2y(t) + D^{\frac{3}{2}}[y(t)] + y(t) = t + 1$; $[D^{\frac{1}{2}-k}h(t)]_{t=0} = 0$, with initial condition $y(0) = 1 = y'(0)$

Solution:

We have $D^2y(t) + D^{\frac{3}{2}}[y(t)] + y(t) = t + 1$

Taking the Rangaig transform of both sides

$\eta\left[D^2[y(t)]\right] + \eta\left[D^{\frac{3}{2}}[y(t)]\right] + \eta[y(t)] = \eta[t] + \eta[1]$;

$\mu^2\Lambda(\mu) + \frac{1}{\mu}h'(0) - h(0) + \mu^{\frac{3}{2}}\left[(-1)^{\frac{3}{2}}\Lambda(\mu) + (-1)^{\frac{3}{2}}\sum_{k=1}^{\frac{3}{2}}(-1)^k\mu^{\frac{3}{2}-(-2-k)}[D^{\frac{3}{2}-k}\,h(t)]_{t=0}\right] + \Lambda(\mu) = -\frac{1}{\mu^3} + \frac{1}{\mu^2}$

we now apply the initial conditions and thus get

$\mu^2\Lambda(\mu) + \frac{1}{\mu} - 1 + \Lambda(\mu) = -\frac{1}{\mu^3} + \frac{1}{\mu^2}$

$\Lambda(\mu)[\mu^2 + 1] + \frac{1}{\mu} - 1 = -\frac{1}{\mu^3} + \frac{1}{\mu^2}$

$\Lambda(\mu) = \left[1 - \frac{1}{\mu^3} + \frac{1}{\mu^2} - \frac{1}{\mu}\right]\left[\frac{1}{\mu^2 + 1}\right]$

$\Lambda(\mu) = \left[\frac{\mu - 1 + \mu^3 - \mu^2}{\mu^3}\right]\left[\frac{1}{\mu^2+1}\right]$

Further simplifying gives

$$\Lambda(\mu) = \left[\frac{(\mu - 1)(\mu^2 + 1)}{\mu^3(\mu^2 + 1)}\right]$$

$$\Lambda(\mu) = \frac{1}{\mu^2} - \frac{1}{\mu^3}$$

To get the required solution now we take the inverse Rangaig transform of both sides

$\eta^{-1}[\Lambda(\mu)] = \eta^{-1}\left[\frac{1}{\mu^2}\right] + \eta^{-1}\left[-\frac{1}{\mu^3}\right]$

We get $y(t) = 1 + t$

4. Conclusion

Thus the Rangaig transform which is relatively new has been used to find solutions to some fractional ordinary differential equations in this paper. It has been shown that this integral transform is relatively straight forward in finding solutions to fractional differential equations. Hence we have seen that the Rangaig transform can be used effectively as a tool to find solutions to ordinary differential equations of fractional order.

4. References

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