Jacobi's Transformation: A Mathematical tool for finding Orthogonal Normal Principal Stress Directions

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Abstract: *Jacobi transformation or a Jacobi rotation is just a plane rotation formed in such a way that by repeated application of this transformation on a real symmetric matrix we can reduce the off diagonal elements to zero. As our stress tensor matrix is also one of the real symmetric matrix and whose latent roots are the Principal Stresses $\sigma_x, \sigma_{y,} \sigma_z$. Principal stress is one kind of maximum or minimum normal stress when the corresponding shear stresses are zero. With the help of Jacobi's method of symmetric matrix we transformed the stress tensor matrix to diagonal matrix to find its latent roots. Therefore major aspect of this paper is to find the orthogonal normal principal stress directions with the technique of Jacobi's transformation. Moreover we can also find the principal normal stresses values along these directions.*

Keywords: Direction Ratios, latent roots, Principal stress, Real symmetric matrix, stress tensor.

1. Introduction

Jacobi's transformation for real symmetric matrix as it is clear from its name is used for symmetric matrix whose entries are real number. This method consists of number of orthogonal similarity transformation. It is one of the easiest, simplest and foolproof techniques to find latent roots of a symmetric matrix. If we know state of stress i.e. all the nine stress component then we can express that in the form of matrix as

$$P = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

In which shear stresses along XY, YZ, ZX planes satisfied the condition that $\tau_{xy} = \tau_{yx}, \tau_{zx} = \tau_{xz}, \tau_{yz} = \tau_{zy}$. Therefore stress tensor matrix is a real symmetric matrix which is the basic condition of Jacobi's transformation.

II. Basic properties

(i) All the latent roots of stress tensor matrix are real i.e. $\bar{\sigma} = \sigma$

(ii) For a stress tensor matrix ∃ a orthogonal matrix $G\ s.t. G^{-1}PG =$ a diagonal matrix

(iii) Diagonal entries of a diagonal matrix are its latent roots.

(iv) Principal stresses are the components of stress tensor when the shear stress components are zero.

III. Methodology

In the given real symmetric matrix, we must select off diagonal element

$$i.e.\ \ Max.\{|\alpha_{ij}|, i \neq j, 1 \leq i \leq n, 1 \leq j \leq n\} = \alpha_{ik},$$

We can construct a submatrix with the entries $\alpha_{ii}, \alpha_{ik}, \alpha_{ki}$ *and* α_{kk}, of order 2 x 2.

$$P = \begin{bmatrix} \alpha_{ii} & \alpha_{ik} \\ \alpha_{ki} & \alpha_{kk} \end{bmatrix}$$

And corresponding to above matrix, we have orthogonal matrix

$$P^* = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Where φ is find in such a way that P is diagonalized.

$$P^{*^{-1}}PP^* = \begin{bmatrix} \alpha_{ii}cos^2\varphi + 2\alpha_{ik}\sin\varphi\cos\varphi + \alpha_{kk}sin^2\varphi & -(\alpha_{ii}-\alpha_{kk})\dfrac{\sin 2\varphi}{2} + \alpha_{ik}\cos 2\varphi \\ -(\alpha_{ii}-\alpha_{kk})\dfrac{\sin 2\varphi}{2} + \alpha_{ik}\cos 2\varphi & \alpha_{ii}sin^2\varphi - 2\alpha_{ik}\sin\varphi\cos\varphi + \alpha_{kk}cos^2\varphi \end{bmatrix}$$

To reduce the above submatrix to diagonal form, we must have

$$\tan 2\varphi = \frac{2\alpha_{ik}}{\alpha_{ii} - \alpha_{kk}}\ , where\ -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$$

With the above value of φ the off diagonal elements vanish while diagonal elements changed accordingly. For the higher order matrix we can construct a transformation matrix as under

$$P^* = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & cos\varphi & \cdots & \cdots & -sin\varphi & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & sin\varphi & \cdots & \cdots & cos\varphi & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

Where all the diagonal elements are unity except two entries $cos\varphi$ in the $(i,i)th\ and\ (k,k)th$ place and all the off diagonal elements be zero except two position occupied by $-sin\varphi\ and\ sin\varphi$ at $(i,k)th\ place\ and\ (k,i)th$ place respectively. If we carry on this process iteratively by selecting largest in magnitude off diagonal element we will finally obtained a diagonal matrix D such that

$$D = Q^{-1}PQ$$

Where Q is product of successive Jacobi rotation matrices from where we can find the orthogonal normal principal directions along which the principal stresses act.

IV. Example: From the stress tensor matrix $P = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$.

Find the Normal Principal stresses values and the orthogonal Normal Principal stress Directions?

Solution: Largest off diagonal element i.e

$$Max.\{|\alpha_{ij}|\, i \neq j, 1 \leq i \leq 3, 1 \leq j \leq 3\} = \alpha_{13} = \alpha_{31} = 2$$

We have $\tan 2\varphi = \frac{2\alpha_{13}}{\alpha_{11}-\alpha_{33}} = \infty$

$$\varphi = \frac{\pi}{4}$$

$$The\ transformation\ matrix\ \ P_1 = \begin{bmatrix} cos\varphi & 0 & -sin\varphi \\ 0 & 1 & 0 \\ sin\varphi & 0 & cos\varphi \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P_1^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P_1^* = P_1^{-1}PP_1 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$In\ P_1^*$, we have largest off diagonal element i.e.

$$Max.\{|\alpha_{ij}|\, i \neq j, 1 \leq i \leq 3, 1 \leq j \leq 3\} = \alpha_{12} = \alpha_{21} = 2$$

$$Now, \tan 2\varphi = \frac{2\alpha_{12}}{\alpha_{11}-\alpha_{22}} = \infty$$

$$\varphi = \frac{\pi}{4}$$

$$The\ transformation\ matrix\ \ P_2 = \begin{bmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2^* = P_2^{-1}PP_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

Therefore Principal Normal stresses are 5,1 and -1.While the directions along which the principal normal stresses acts can also be determine as from the product of Jacobi's transforming matrix as under

$$P^* = P_1P_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Columns of the above matrix provided the information regarding the direction of the Principal normal stresses. Therefore the direction along which Principal Normal stresses 5,1 and -1 acts has direction ratio $< \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2} >, < -\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2} > and\ < -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} >$ respectively. Now, If there Principal stresses directions are inclined at an angle $\alpha, \beta\ and\ \gamma$ respectively.

We have $\cos\alpha = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = 0$

Similarly $\cos\beta = \cos\gamma = 0$

Therefore $\alpha = \beta = \gamma = \frac{\pi}{2}$.

Conclusion:

Principal stresses directions are very helpful in material designing. Performing a stress analysis is a great way for members of the engineering and manufacturing industries to ensure their final product is reliable. Also it helps you identify where your structures are meeting expectations and how you can improve on others.

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