Lie Symmetries and Classifications of (2+1) – Dimensional Potential Huxley Equation

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Abstract

The (2+1) - dimensional potential Huxley equation $u_t - u^2_x - u_{xx} - u_{yy} - (k - u)(u - 1)u = 0$ is considered. A symmetry classification of the equation using lie group method is presented and reduction to the first- or second – order ordinary differential equation is provided.

Keywords: Abelian subalgebras, Lie bracket *,Lie group, Symmetry Classification , Two-dimensional nonlinear potential Huxley equation.*

1. Introduction:

The one - dimensional heat equation is extensively studied from the point of view of its Lie point symmetries. A detailed symmetry analysis of this equation can be found in P.J. Olver [11], Cantwell [1], Ibragimov [2] and Bluman and Kumei [3] .Since thermal diffusivity of some materials may be a function of temperature, it introduces nonlinearities in the heat equation that models such phenomenon. This shows that whereas nonlinear heat equation models real world problems the best, it may be difficult to tackle such problems by usual methods. In an attempt to study nonlineareffects Saied and Hussain [4] gave some new similarity solutions of the (1+1) - nonlinear heat equation. Later Clarkson and Mansfield [5] studied classical and non-classical symmetries of the (1+1)-heat equation and gave new reductions for the linear heat equation and a catalogue of closed form solutions. In higher dimensions Servo [6] gave some conditional symmetriesfor a nonlinear heat equation while Goardetal. [7] studied the nonlinear heat equation in the degenerate case. Nonlinear heat equations in one or higher dimensions are also studied in literature by using both symmetry as well as other methods [8, 9]. An account of some interesting cases is given by Polyanin [10].

In this paper, we discuss the Lie symmetry and classifications of the (2+1) - dimensional potential Huxley equation

$$u_t - u^2_x - u_{xx} - u_{yy} - (k - u)(u - 1)u = 0 \qquad (1.1)$$

Our intention is to show that equation (1.1) admits a two-dimensional symmetry group and determine the corresponding Lie algebra, classify the one- and two-dimensional sub-algebras of the symmetry algebra of equation (1.1) in order to reduce equation (1.1) to (1+1) - dimensional PDEs and then to ODEs. It is shown that equation (1.1) reduces to a once differentiated Potential Huxley equation. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [12] to successively reduce equation (1.1) to (1+1) - dimensional PDEs and ODEs with the help of two-dimensional abelian and non-abelian solvable subalgebras.

This paper is organised as follows: In section 2, we determine the symmetry group of equation (1.1) and write down the associated Lie algebra. In section 3, we consider all one - dimensional subalgebras and obtain the corresponding reductions to (1+1) - dimensional PDEs. In section 4, we show that the generators form a closed Lie algebra and use this fact to reduce equation (1.1) successively to (1+1) - dimensional PDEs and ODEs. In section 5, we summarises the conclusions of the present work

2.**The Symmetry Group and Lie Algebra of $u_t - u^2_x - u_{xx} - u_{yy} - (k - u)(u - 1)u = 0$**:

If equation (1.1) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [3], Olver [11])

$x^* = x + \epsilon\, \xi(x, y, t; u) + O(\epsilon^2),$ (2.1)

$y^* = y + \epsilon\, \eta(x, y, t; u) + O(\epsilon^2),$ (2.2)

$t^* = t + \epsilon\, \tau(x, y, t; u) + O(\epsilon^2),$ (2.3)

$u^* = u + \epsilon\, \varphi(x, y, t; u) + O(\epsilon^2),$ (2.4)

with infinitesimal generator

$X = \xi(x, y, t; u)\frac{\partial}{\partial x} + \eta(x, y, t; u)\frac{\partial}{\partial y} + \tau(x, y, t; u)\frac{\partial}{\partial t} + \varphi(x, y, t; u)\frac{\partial}{\partial u},$ (2.5)

then the invariant condition is

$\varphi_t + \varphi^2{}_x - (\varphi_{xx} + \varphi_{yy}) - (k - \varphi)(\varphi - 1)\,\varphi = 0.$ (2.6)

In order to determine the four infinitesimal ξ, η, τ and φ, we prolong V tosecond order. This prolongation is given by the formula

$V^{(2)} = v + \varphi^x\frac{\partial}{\partial u_x} + \varphi\frac{\partial}{\partial u_y} + \varphi^t\frac{\partial}{\partial u_t} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^{xy}\frac{\partial}{\partial u_{xy}} + \varphi^{xt}\frac{\partial}{\partial u_{xt}} + \varphi^{yy}\frac{\partial}{\partial u_{yy}} + \varphi^{yt}\frac{\partial}{\partial u_{yt}} + \varphi^{tt}\frac{\partial}{\partial u_{tt}}.$ (2.7)

In the above expression every co-efficient of the prolonged generator is a function of x, y, t and u can be determined by the formulae,

$\varphi^i = D_i(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i},$ (2.8)

$\varphi^{ij} = D_iD_j(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij},$ (2.9)

where D_i represents total derivative and subscripts of u derivative with respect to the respective co-ordinates. To proceed with reductions of equation (1.1) we now use symmetry criterion for PDEs. For given equation this criterion is expressed by the formula.

$V^{(2)}[\, u_t - u^2{}_x - u_{xx} - u_{yy} - (k - u)(u - 1)u] = 0.$

In we introduced the following quantities

$\varphi^t = D_t(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,t} + \eta u_{y,t} + \tau u_{t,t}$

$= \varphi_u + u_{t,}(\varphi_u - \tau_t) - \tau_u u^2{}_t - \xi_u u_t u_x - \xi_t u_x - \eta_u u_t u_y - \eta_u u_y.$

$\varphi^x = D_x(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{yx} + \tau u_{tx}$

$= \varphi_x + (\varphi_u - \xi_x)u_x - \eta_x u_y - \tau_x u_t - \xi_u u^2{}_x - \eta_u u_x u_y - \tau_u u_x u_t$

$\varphi^{xx} =$
$D_X D_X(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxx} + \eta u_{yxx} + \tau u_{txx}$
$= \varphi_{xx} + (2\varphi - \xi_{xx})u_x - \eta_{xx}u_y - \tau_{xx}u_t + (\varphi_u - 2\xi_x)u_{xx} - 2\eta_x u_{xy} - 2\tau_x u_{xt} + (\varphi_{uu} - 2\xi_{ux})u^2{}_x - 2\eta_{ux}u_x u_y - 2\tau_{xu}u_x u_t -$
$\xi_{uu}u^3{}_x - 3\xi_u u_x u_{xx} - \eta_{uu}u^2{}_x u_y - \tau_{uu}u^2{}_x u_t - 2\eta_u u_x u_{xy} - \eta_u u_{xx}u_y - \tau_u u_{xx}u_t - 2\tau_u u_x u_{xt}$
$\varphi^{yy} = D_y D_y(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxx} + \eta u_{yxx} + \tau u_{txx}$
$= \varphi_{yy} - \xi_{yy}u_x + (2\varphi_{yu} - \eta_{yy})u_y - \tau_{yy}u_t - 2\xi_y u_{xy} + (\varphi_u - 2\eta_y)u_{yy} - 2\xi_{yu}u_x u_y - 2\tau_{yu}u_y u_t + (\varphi_{uu} - 2\eta_{yu})u^2{}_y - 2\tau_y u_{yt} -$
$2\xi_u u_y u_{xy} - 3\eta_u u_y u_{yy} - \xi_{uu}u^2{}_y u_x - \xi_u u_{yy}u_x - \eta_{uu}u^2{}_y - 2\tau_u u_y u_{yt} - \tau_u u_{yy}u_t - \tau_{uu}u^2{}_y u_t$.

Substitute them in equation (2.6) and then compare co- efficients of various monomials in derivatives of 'u'. This yields the following equations:

$\xi_u = 0,$
$\eta_u = 0,$
$\tau_u = 0,$
$\varphi_u = 0,$
$\xi_y = 0,$
$\tau_y = 0,$
$\eta_x = 0,$
$\tau_x = 0,$
$-\tau_t - 2u\varphi_x + \xi_{xx} - 2\varphi_{xu} = 0,$
$K\varphi - 2u\varphi - 2ku\varphi + 3u^2\varphi + ku\tau_t - u^2\tau_t - ku^2\tau_t + u^3\tau_t + \varphi_t - ku\varphi_u + u^2\varphi_u + ku^2\varphi_u - u^3\varphi_u - \varphi_{xx} - \varphi_{yy} = 0,$
$2\xi_x - \tau_t - \varphi_u = 0,$
$2\xi_x - \tau_t = 0,$
$-\eta_t + \eta_{yy} - 2\tau\varphi_{yu} = 0,$
$2\eta_y - \tau_t = 0.$ (2.10)

Using the above equations and some more manipulations, we get,

$\xi = k_1,$ (2.11)
$\eta = k_2,$ (2.12)
$\tau = k_3,$ (2.13)
$\varphi = 0$. (2.14)

At this stage, we construct the symmetry generators corresponding to eachof the constants involved. These are a total of four generators given by,

$V_1 = \frac{\partial}{\partial x}$,
$V_2 = \frac{\partial}{\partial y}$,
$V_3 = \frac{\partial}{\partial t}$. (2.15)

The one-parameter groups $g_i(\epsilon)$ generalized by the V_i, where I = 1, 2 and 3 are

$g_1(\epsilon) : (x, y, t; u) \rightarrow (x + \epsilon, y, t, u)$,
$g_2(\epsilon) : (x, y, t; u) \rightarrow (x, y+ \epsilon, t, u)$,
$g_3(\epsilon) : (x, y, t; u) \rightarrow (x, y, t+\epsilon, u)$.

Where $\exp(\epsilon V_i)$ (x, y, t; u) = $(\bar{x}, \bar{y}, \bar{t}; \bar{u})$ and
(i) g_3 is time translation ,
(ii) g_1 and g_2 are the space-invariant of the equation. The symmetry generators found in equation (2.15) form a closed Lie Algebra whose commutation table is shown below.

Table 1

Commutation relations satisfied by above generators is

$[V_i,V_j]$	V_1	V_2	V_3

V_1	0	0	0
V_2	0	0	0
V_3	0	0	0

The commutation relations of the Lie algebra, determined by V_1,V_2 and V_3 are shown in the above table.

For this two-dimensional Lie algebra the commutator table for V_i is a $(3\otimes 3)$ table whose $(i,j)^{th}$ entry expresses the Lie Bracket $[V_i,V_j]$ given by the above Lie algebra L. The table is skew-symmetric and the diagonal elements all vanish.

It is evident from the commutator table that there are all two-dimensional abelian subalgebras, namely,

$L_A,1 = \{V_1,V_2\}$, $L_A,2 = \{V_1,V_3\}$, $L_A,3 = \{V_2,V_3\}$.

3. Reductions of $u_t - u^2_x - u_{xx} - u_{yy} - (k - u)(u - 1)u = 0$ by One-Dimensional Subalgebras:

Case 1 : $V_1 = \partial_x$

The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}$$

Following standard procedure we integrate the characteristic equation to getthree similarity variables,

$y = s$, $t = r$ and $u = w(r,s)$ (3.1)

Using these similarity variables in equation (1.1) can be recast in the form

$w_r - w_{ss} - (k - w)(w - 1)w = 0$ (3.2)

Case 2 : $V= \partial_y$

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}$$

Following standard procedure we integrate the characteristic equation to getthree similarity variables,

$x = s$, $t = r$ and $u = w(r,s)$ (3.3)

Using these similarity variables in equation (1.1) can be recast in the form

$w_r - w_s^2 - w_{ss} - (k - w)(w - 1)w = 0.$ (3.4)

Case 3 : $V_3 = \partial_t$

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}$$

Following standard procedure we integrate the characteristic equation to getthree similarity variables,

$x = s$, $y = r$ and $u = w(r,s)$ (3.5)

Using these similarity variables in equation (1.1) can be recast in the form

$W_s^2 - (w_{ss} + w_{rr}) - (k - w)(w - 1)w = 0.$ (3.6)

4. Reductions of $u_t - u^2_x - u_{xx} - u_{yy} - (k - u)(u - 1)u = 0$ by Two - Dimensional Subalgebras:

Case I : Reduction under V_1 and V_2.

From Table 1 we find that the given generators commute $[V_1,V_2] = 0$.

Thus either of V_1or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. Therefore we get equation (3.1) and equation (3.2).

At this stage, we express V_2 in terms of the similarity variables defined in equation (3.1).

The transformed V_2 is

$\tilde{V}_2 = \partial_s$.

The characteristic equation for $\tilde{V}_2$ is

$$\frac{ds}{1} = \frac{dr}{0} = \frac{dw}{0}$$

Integrating this equation as before leads to new variables r = α and w = $\beta(\alpha)$,

which reduce equation (3.2) to a second-order ODE

$\beta' - (k - \beta)(\beta - 1)\beta = 0.$ (4.1)

Case II : Reduction under V_1 and V_3.

From Table 1 we find that the given generators commute $[V_1,V_3] = 0$.

Thus either of V_1 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_1. Therefore we get equation (3.5) and equation (3.6).

At this stage, we express V_3 in terms of the similarity variables defined in equation (3.3).

The transformed V_3 is

$\tilde{V}_3 = \partial_r$.

The characteristic equation for $\tilde{V}_3$ is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0}$$

Integrating this equation as before leads to new variable

s = α and w = $\beta(\alpha)$,

which reduce equation (3.2) to a second-order ODE

$\beta'' + (k - \beta)(\beta - 1)\beta = 0.$ (4.2)

Case III : Reduction under V_2 and V_3 .

In this case the two symmetry generators V_2 and V_3 satisfy the commutation relation $[V_2,V_3] = 0$. Thus either of V_2 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_2. Therefore we get equation (3.5) and equation (3.6).

At this stage, we express V_3 in terms of the similarity variables defined in equation (3.5).

The transformed V_3 is

$\tilde{V}_3 = \partial_r$.

The characteristic equation for $\tilde{V}_3$ is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0}$$

Integrating this equation as before leads to new variable

s = α and w = $\beta(\alpha)$,

which reduce equation (3.2) to a second-order ODE

$\beta'^2 + \beta'' + (k - \beta)(\beta - 1)\beta = 0.$ (4.3)

Table 2

Algebra	Reduction
$[V_1,V_2] = 0$	$\beta' - (k - \beta)(\beta - 1)\beta = 0$
$[V_1,V_3] = 0$	$\beta'' + (k - \beta)(\beta - 1)\beta = 0$
$[V_2,V_3] = 0$	$\beta'^2 + \beta'' + (k - \beta)(\beta - 1)\beta = 0$

5. Conclusions

In this Chapter,

1) A (2+1) - dimensional Potential Huxley equation $u_t - u^2_x - u_{xx} - u_y - (k - u)(u - 1)u = 0$ is subjected to Lie's classical method.
2) Equation (1.1) admits a Two-dimensional symmetry group.
3) It is established that the symmetry generators form a closed Lie algebras.

4) Classification of symmetry algebra of equation (1.1) into one- and two-dimensional subalgebras is carried out.
5) Systematic reduction to (1+1) - dimensional PDE and then to first orsecond order ODEs are performed using one-dimensional and two-dimensional solvable abelian subalgebras.

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References:

[1] B.J. Cantwell, An Introduction to Symmetry Analysis, Cambridge University Press, Cambridge, 2002.

[2] N.H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, John Wiley, New York, 1999.

[3] G. Bluman, S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York, 1989.

[4] E.A. Saied, M.M. Hussain, Similarity solutions for a nonlinear model ofthe heat equation, J. Nonlinear Math. Phys. 3 (1-2) (1996) 219-225.

[5] P.A. Clarkson, E.L. Mansfield, Symmetry reductions and exact solutionsof a class of nonlinear heat equations, Phys. D 70 (1993) 250-288.

[6] M.I. Servo, Conditional and nonlocal symmetry of nonlinear heat equation, J.Nonlinear Math. Phys. 3 (1-2) (1996) 63-67.

[7] J.M. Goard, P. Broadbridge, D.J. Arrigo, The integrable nonlinear degenerate diffusion equations, Z. Angew. Math. Phys. 47 (6) (1996) 926-942.

[8] P.G. Estevez, C. Qu, S.L. Zhang, Separation of variables of a generalized porous medium equation with nonlinear source, J. Math. Anal. Appl. 275(2002) 44-59.52

[9] P.W. Doyle, P.J. Vassiliou, Separation of variables in the 1-dimensional non -linear diffusion equation, Internat. J. Non-Linear Mech. 33 (2)(2002) 315-326.

[10] A.D. Polyanin, Handbook of Linear Partial Differential Equations for Engineers and Scientists, CRC, Boca Raton, 2002.

[11] P.J. Olver, Applications of Lie Groups to Differential Equations,Springer-Verlag, New York, 1986.

[12] A. Ahmad, Ashfaque H. Bokhari, A. H. Kara, F. D. Zaman, Symmetry Classifications and Reductions of Some Classes of (2+1)-Nonlinear Heat Equation, J. Math. Anal. Appl., 339: 175-181(2008).

[13] Z. Liu and C. Yang ,The application of bifurcation method to a higher-order KdV equation, J. Math.Anal. Appl., 275: 1-12 (2002).

[14] R. M. Miura, Korteweg-de Vries equations and generalizations. A remarkable explicit nonlinear transformation, I.Math. Phys. 9: 1202.1204(1968).

[15] M. Senthilkumaran, D. Pandiaraja and B. MayilVaganan, New Exactexplicit Solutions of the Generalized KdV Equations, 2008 Appl. Math.comp. 202: 693-699.

[16] G. B. Whitham, Linear and Nonlinear Waves, Wiley, New York,1974.

[17] D. J. Korteweg and G. de Vries, On the Chans of Form of Long Waves Advancing in a Rectangular canal, and On a New type of Long Stationary Waves, Philosophical Magazine 39 (1985) 422-443.

[18] F. Gungor and P. Winternitz, Generalized KadomtsevPetviashvili equation with an infinitesimal dimensional symmetry algebra, J. Math. Anal.276 (2002), 314-328.

[19] F. Gungor and P. Winternitz, Equaivalence Classes and Symmetries of the Variable Coefficient KadomtsevPetviashvili Equation, Nonlinear Dynamics 35 (2004) 381-396.

[20] B. MayilVaganan and N. Muthumari, Symmetry Classifications and Reduction, International Journal ofApplied Mathematics, Vol 24 (5): (2011) 653-662.