LIE SYMMETRIES AND REDUCTIONS OF (2+1) DIMENSIONAL MODIFIED EQUAL WIDTH WAVE EQUATION

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Abstract

Numerical solution of the modified equal width wave equation is obtained by using lumped Galerkin method based on cubic B-Spline finite element method. Solitary wave motion and interaction of two Solitary waves are studied using the proposed method. The (2+1) non-linear wave equation $u_t+u^3u_x+u^3u_y-u_{xxt}-u_{yyt} = 0$ is considered. A symmetry classification of the equation using Lie group method is presented and reduction to the first or second order ordinary differential equation is provided.

Keywords

Korteweg and de vries (KdV) equation, two dimensional non-linear wave equation, symmetry classifications.

1. Introduction

In the last few decades, the traditional integral transform methods such as Fourier and Laplace transform have commonly been used to solve the engineering problems. These methods transform the differential equations into algebraic equations which are easier to deal with.

The well – known Korteweg and de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0$$

is a non-linear partial differential equations (PDE) that models the time independent motion of shallow water waves in one space dimension. Morrison et al [1] proposed the one dimensional PDE

$$u_t + uu_x - \mu u_{xxt} = 0$$

as an equally valid and accurate model for the same wave phenomena simulated by the KdV equation. This PDE is called the equal width wave equation because the solutions for solitary waves with a permanent form and speed, for a given value of the parameter μ are waves with an equal width or wave length for all wave amplitudes.

The equal width (EW) was suggested by Morrison et al [1], to use as a model partial differential equation for the stimulation of one-dimensional wave propagation in non-linear media with dispersion processes. Many methods, have been proposed to solve the EW equation [2-5]. Based on the EW equation, Zaki [6] considered the solitary wave interaction for the modified equal width (MEW) equation by Petrov – Galerkin method using quintic. B – Spline finite elements, Wazwaz [7] investigated the modified equal width wave equation and two of its variants by the tanh and the sine – cosine methods and saka [8] proposed algorithms for the numerical solutions of the modified equal width wave equation usind collocation method. Reviewing theses improvements, we can find that its inevitable to involve the quintic. B – Splines, the computations of time dependent parameters, linearization and discretization of the MEW equation, while the tanh and sine – cosine methods are based on the solutions that can be expressed in terms of the tahn, sechor each functions. Therefore, these methods for solving the MEW equation narrow down their applications. A nature question kept in our mind is that whether we can solve the MEW equation without quintic B – Splines, linearization and discretization. Also Lu [9] applied variational iteration method for solving the modified equal width wave equation.

2. Symmetries and classifications of lie algebra for $u_t + u^3u_x + u^3u_y - u_{xxt} + u_{yyt} = 0$

In order to derive the symmetry generators of Eqn. (1) and obtain the closed form solutions for all f(u), we consider one parameter Lie point transformation that leaves (1) invariant. This transformation is given by

$$\tilde{x}^i = x^i + \varepsilon\xi^i(x,y,t,u) + O(\varepsilon^2), i = 1,2,....4 \quad (2)$$

Where $\xi^i = \frac{\partial \tilde{x}^i}{\partial \varepsilon}|_{\varepsilon=0}$ defines the symmetry generator associated with (2) given by

$$V = \xi\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial x^i} + \tau\frac{\partial}{\partial x^i} + \phi\frac{\partial}{\partial x^i} \quad (3)$$

In order to determine four components ξ^i, we prolong V to third order. This prolongation is given by the formula

$$V^{(3)} = V + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}$$
$$+\phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxy}\frac{\partial}{\partial u_{xxy}} + \phi^{xxt}\frac{\partial}{\partial u_{xxt}} + \phi^{yyy}\frac{\partial}{\partial u_{yyy}} + \phi^{yyt}\frac{\partial}{\partial u_{yyt}} \quad (4)$$

In above expression every coefficient of the prolonged generator is a functions of (x,y,t,u) and can be determined by the formulae

$$\phi^i = D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i} \quad (5)$$

$$\phi^{ij} = D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij} \quad (6)$$

Where D_i represents total derivative and subscripts of *u* derivative with respect to the respective coordinates. To proceed with reductions of (1) we now use symmetry criterion for partial differential equations. For heat equation this criterion is expressed by the formula

$$V^{(3)}[u_t + u^3u_x + u^3u_y - u_{xxt} - u_{yyt}] = 0$$

Whenever,

$$u_t + u^3u_x + u^3u_y = u_{xxt} + u_{yyt}$$

Using the symmetry criterion with Eqn.(4) in mind immediately yields

$$\phi^t + \phi^3\phi^x + \phi^3\phi^y - \phi^{xxt} - \phi^{yyt} = 0. \quad (7)$$

At this stage we calculate expression for $\phi^t, \phi^x, \phi^y, \phi^{xxt}, \phi^{yyt}$ using (5)-(6), substitute them in (6) and then compare coefficients of various monomials in derivative of u. this yields the following system of over-determined partial differential equations:

$$\xi_u = \eta_u = \tau_u = 0$$
$$\phi_{uu} = 0$$
$$\xi_t = \eta_t = 0$$
$$\phi_{tu} = 0$$
$$\tau_y = \tau_x = 0$$
$$\phi_t + u^3\phi_x + u^3\phi_y - \phi_{xxt} - \phi_{yyt} = 0$$
$$3u^2\phi - u^3\xi_x - u^3\xi_y + u^3\tau_t - 2\phi_{xtu} + u^3\phi_{xxu} + u^3\phi_{yyu} = 0$$
$$3u^2\phi - u^3\eta_x - u^3\eta_y + u^3\tau_t - 2\phi_{xty} + u^3\phi_{xxu} + u^3\phi_{yyu} = 0$$
$$2\xi_x - \phi_{xxu} - \phi_{yyu} = 0$$
$$2\eta_y - \phi_{xxu} - \phi_{yyu} = 0$$
$$\eta_{xx} + \eta_{yy} - 2\phi_{yu} = 0$$
$$\xi_{xx} + \xi_{yy} - 2\phi_{xu} = 0$$
$$\eta_x + \xi_y = 0$$

3. Reduction of one dimensional Abelian Sub-algebra for $u_t + u^3u_x + u^3u_y - u_{xxt} + u_{yyt} = 0$

After some more manipulations one finds that $\eta\, and\, \xi$ becomes

$$\xi = k_4 \qquad \eta = -k \tag{8}$$

The remaining equations can then be used to determine τ and φ as

$$\tau = k_1 + k_2 t \qquad \phi = -\frac{k_2 u}{3} \tag{9}$$

At this stage we construct the symmetry generators corresponding to each of the constants involved. These are a total of eight generators given by

$$V_1 = \partial t \qquad V_2 = u\partial u \qquad V_3 = \partial y \qquad V_4 = \partial x \tag{10}$$

It is easy to check that the symmetry generators found in (10) form a closed Lie algebra whose communication relations are given in Table 1

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	0

V_2	0	0	0	0
V_3	0	0	0	0
V_4	0	0	0	0

Commutation relations satisfied by generators

4. Reduction of two dimensional Abelian Sub-algebra for $u_t + u^3u_x + u^3u_y - u_{xxt} + u_{yyt} = 0$

We now briefly show steps involved in the reduction of the nonlinear heat equation to a second-order differential equation. Since reduction under all the sub-algebras cannot be given in the paper, we restrict ourselves to giving reductions in two cases only, i.e., $[V_1,V_3]$ and $[V_2,V_4]$. Reduction in the remaining cases is listed in the form of Appendices A at the end of the paper.

4.1 Reduction under V_1 and V_3

From Table 1 we find that the given generators commute $[V_1,V_3]=0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our purpose we begin reduction with V_1. The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}.$$

Following standard procedure we integrate the characteristics equation to get three similarity variables.

$$s = y, \quad x = r, \quad u = w(r,s) \qquad (11)$$

Using these similarity variables Eqn. (A) can be recast in the form

$$w_r + w_s = 0 \qquad (12)$$

At this stage we express V_3 in terms of the similarity variables defined in Eqn.(11). It is straight forward to note that V_3 in the new variables takes the form

$$\tilde{V}_3 = \frac{\partial}{\partial s} \qquad (13)$$

The characteristics equation for $\tilde{V}_3$ is,

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}$$

Integrating this equation as before leads to new variables $\xi = r$ and $R(\xi) = w$, which reduces (12) to a second-order differential equation

$$R' = 0 \qquad (14)$$

4.2 Reduction under V_2 and V_4

In this case the two symmetry generators V_2 and V_4 satisfy the communication relation $[V_2,V_4] = 0$. This suggests that reduction in this case should start with V_3. The similarity variables are

$$s = y, \quad x = r, \quad u = \frac{w(r,s)}{t}$$

and K is a constant. The corresponding reduced partial differential equation is

$$-w + t\,(w^3w_r + w^3w_s) + w_{rr} + w_{ss} = 0 \qquad (15)$$

The transformed V_1 is

$$\vec{V}_4 = \frac{\partial}{\partial r} \quad (16)$$

The invariants of $\vec{V}_4$ are

$$\xi = s \text{ and } R(\xi) = w$$

Which reduce Eqn. (15) to the ordinary differential equation

$$R'' + R'R^3 t - R = 0 \quad (17)$$

Reductions in remaining cases using generators forming sub-algebra are given in the form of Table 2 in Appendix A.

Appendix A

Algebra	Reduction
$[V_1,V_2]$	$R' = 0$
$[V_1,V_3]$	$R' = 0$
$[V_1,V_4]$	$R' = 0$
$[V_2,V_3]$	$R'' + R'R^3 t - R = 0$
$[V_2,V_4]$	$R'' + R'R^3 t - R = 0$
$[V_3,V_4]$	$R' = 0$

5. CONCLUSION

In this chapter,

i) A (2+1) dimensional KDV equation $u_t + u^3u_x + u^3u_y - u_{xxt} - u_{yyt} = 0$, is subjected to Lie's classical method.

ii) Equation $u_t + u^3u_x + u^3u_y - u_{xxt} - u_{yyt} = 0$ admits a four dimensional symmetry group.

iii) It is established that the symmetry generators form a closed Lie algebra.

iv) Classification of symmetry algebra of $u_t + u^3u_x + u^3u_y - u_{xxt} - u_{yyt} = 0$ into one and two dimensional sub-algebras is carried out.

v) Systematic reduction to (1+1) – dimensional PDE and then to first or second order ODEs are performed using one-dimensional and two-dimensional solvable abelian sub-algebras.

6. REFERENCES

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