ON DOUBLE REPRESENTATION OF QUATERNION QUASI-NORMAL MATRICES

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ABSTRACT

In this paper, the properties of quaternion quasi-normal matrices in the form of double representation of complex matrices. The normal product of the quaternion quasi-normal matrices are derived.

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INTRODUCTION:

A normal matrix $A = (a_{ij})$ with complex elements is a matrix such that $AA^{CT} = A^{CT}A$ where A^{CT} denotes the (complex) conjugate transpose of A. In an article by K. Morita[5] a quasi-normal matrix is defined to be a complex matrix A which is such that $AA^{CT} = A^T A^C$, where T denotes the transpose of A and A^C the matrix in which each element is replaced by its conjugate, and certain basic properties of such a matrix are developed there.

Based on the bi-complex form of quaternion matrix Junliang Wu and Pingping Zhang [4] presented some new concept to quaternion division algebra. The new concepts could perfect the theory of Wu in [9]. The complex representation method for the quaternion matrices on explore the relation between the quaternion matrices and complex matrices.

In this paper, quaternion quasi-normal matrix is defined. The further properties of quaternion quasi-normal are developed, their relation in a sense, to a quaternion normal matrices are consider and further results concerning quaternion normal products are obtained for quaternion quasi-normal.

Theorem: 1

A matrix A is double representation of quaternion quasi-normal iff a quaternion unitary matrix U such that UAU^T is a direct sum of non-negative real numbers and of 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a and b are non-negative real numbers.

Proof:

Let A be a double representation of quaternion quasi-normal where $A = X + Y$. [Where $X = X_0 + X_1 j$ and $Y = Y_0 + Y_1 j$],Where $X = X^T = X_0^T + X_1^T j$ and $Y = -Y^T = -(Y_0^T + Y_1^T j)$. Then $AA^{CT} = A^T A^C$ where $A = X + Y$.

$$\begin{aligned} AA^{CT} &= (X+Y)(X+Y)^{CT} \\ &= [(X_0+Y_0)+(X_1+Y_1)j][(X_0+X_1 j)+(Y_0+Y_1 j)]^{CT} \\ &= [(X_0+Y_0)+(X_1+Y_1)j][(X_0+X_1 j)^{CT}+(Y_0+Y_1 j)^{CT}] \\ &= [(X_0+Y_0)+(X_1+Y_1)j][(X_0^C - X_1^C j)+(-Y_0^C+Y_1^C j)] \end{aligned}$$

$$= [(X_0+Y_0)+(X_1+Y_1)j][(X_0^C-Y_0^C)-(X_1^C-Y_1^C)j]$$
$$= [(X_0+Y_0)(X_0^C-Y_0^C)]-[(X_1+Y_1)(X_1^C-Y_1^C)j]$$
$$= [X_0X_0^C-X_0Y_0^C+Y_0X_0^C-Y_0Y_0^C]+[X_1Y_1^C-X_1X_1^C-Y_1X_1^C+Y_1Y_1^C]j \quad ......(1)$$

Now,

$$A^TA^C = (X+Y)^T(X+Y)^C$$
$$= (X_0+X_1j+Y_0+Y_1j)^T\,(X_0+X_1j+Y_0+Y_1j)^C$$
$$= [(X_0+Y_0)+(X_1+Y_1)j]^T[(X_0+X_1j)+(Y_0+Y_1j)]^C$$
$$= [(X_0^T+Y_0^T)+(X_1^T+Y_1^T)j][(X_0^C+Y_0^C)-(X_1^C+Y_1^C)j]$$
$$= [(X_0^T+X_1^Tj)+(Y_0^T+Y_1^T)j][(X_0^C+Y_0^C)-(X_1^C+Y_1^C)j]$$
$$= [(X_0-Y_0)(X_0^C+Y_0^C)]-[(X_1-Y_1)(X_1^C+Y_1^C)]j$$
$$= [X_0X_0^C+X_0Y_0^C-Y_0X_0^C-Y_0Y_0^C]+[Y_1X_1^C+Y_1Y_1^C-X_1X_1^C-X_1Y_1^C]j \quad ......(2)$$

Since A is double representation of quaternion quasi-normal.

$$AA^{CT} = A^TA^C$$
$$[X_0X_0^C-X_0Y_0^C+Y_0X_0^C-Y_0Y_0^C]+[X_1Y_1^C-X_1X_1^C-Y_1X_1^C+Y_1Y_1^C]j = [X_0X_0^C+X_0Y_0^C-Y_0X_0^C-Y_0Y_0^C]+[Y_1X_1^C+Y_1Y_1^C-X_1X_1^C-X_1Y_1^C]j$$
$$Y_0X_0^C+Y_0X_0^C+X_1Y_1^Cj+X_1Y_1^Cj = X_0Y_0^C+X_0Y_0^C+Y_1X_1^Cj+Y_1X_1^Cj$$
$$2Y_0X_0^C+2X_1Y_1^Cj = 2X_0Y_0^C+2Y_1X_1^Cj$$
$$Y_0X_0^C-Y_1X_1^Cj = X_0Y_0^C-X_1Y_1^Cj$$
$$(Y_0+Y_1j)(X_0^C-X_1^Cj) = (X_0+X_1j)(Y_0^C-Y_1^Cj)$$
$$YX^C = XY^C$$

There exists a quaternion unitary matrix $U=U_0+U_1j$ such that U_0+U_1j [7], UXU^T = $(U_0X_0U_0^T)+(U_1X_1U_1^T)$ = D is a diagonal matrix with non-negative real. Therefore,

$$(UYU^T)(UXU^T)^C = (UXU^T)(UYU^T)^C$$
$$U_0Y_0U_0^TU_0^CX_0^CU_0^{TC}-U_1Y_1U_1^TU_1^CX_1^CU_1^{TC}j = U_0X_0U_0^TU_0^CY_0^CU_0^{TC}-U_1X_1U_1^TU_1^CY_1^CU_1^{TC}j$$

Or $WD = DW^C$, where $W=-W^T$. Let U_0+U_1j be chosen so that D is such that $d_s \ge d_t \ge 0$ for $s<t$ where d_s is the s^{th} diagonal element of D.

If $W = (e_{st})$ where $e_{ts} = -e_{st}$, then $e_{st}d_t = d_t\bar{e}_{st}$ for $t>s$ and three possibilities may occur: if $d_s = d_t \neq 0$, the e_{st} is real; if $d_s = d_t = 0$, e_{st} is arbitrary (though $W=-W^T$ still holds); and if $d_s \neq d_t$, then $e_{st} = 0$ for if $e_{st} = a+ib$ then $(a+ib)d_t = d_s(a-ib)$ and $a(d_t-d_s) = 0$ implies that $a=0$ and $b(d_s+d_t) = 0$ implies that $d_s = -d_t$ (which is not possible since d_s are real and non-negative and $d_s \neq d_t$) or $b = 0$ so $e_{st} = 0$.

So if $UXU^T = (U_0X_0U_0^T)+(U_1X_1U_1^T)j = d_1I_1\oplus d_2I_2\oplus........\oplus d_kI_k$ where $\oplus$ denotes the direct sum, then $UYU^T = (U_0Y_0U_0^T)+(U_1Y_1U_1^T)j = Y_1\oplus Y_2\oplus......\oplus Y_k$ where $Y_s=-Y_s^T$ is real and $Y_k=-Y_k^T$ is quaternion if and only if $d_k = 0$. For each real Y_s there exists a real orthogonal matrix V_s so that $V_s Y_s V_s^T$ is a direct sum of zero matrices and matrices of the form $\begin{bmatrix} 0 & b_1 + b_2 j \\ -b_1 - b_2 j & 0 \end{bmatrix}$ where b_1 and b_2 are real.

If $Y_k = (Y_{0(k)} + Y_{1(k)} j) = -(Y_{0(k)} + Y_{1(k)} j)^T$ is quaternion, there exists a quaternion unitary matrix $V_k = V_{0(k)} + V_{1(k)} j$ such that $V_{0(k)} Y_{0(k)} V_{0(k)} + V_{1(k)} Y_{1(k)} V_{1(k)} j$, $Y_0 + Y_1 j$ is a direct sum of matrices of the some form, so that if $V = V_1 \oplus V_2 \oplus ..... \oplus V_k$, then $V = V_0 + V_1 j$, then $VUXU^T V^T = (V_0 U_0 X_0 U_0^T V_0^T) + (V_1 U_1 X_1 U_1^T V_1^T) j = D$ and $VUYU^T V^T = (V_0 U_0 Y_0 U_0^T V_0^T) + (V_1 U_1 Y_1 U_1^T V_1^T) j = F$ the direct sum described. Therefore, $VUAU^T V^T = VU(X+Y)U^T V^T = [(V_0 U_0 X_0 U_0^T V_0^T) + (V_1 U_1 X_1 U_1^T V_1^T) j] + [(V_0 U_0 Y_0 U_0^T V_0^T) + (V_1 U_1 Y_1 U_1^T V_1^T) j] = D + F$ Which is the desired form.

Properties of double representation of quaternion quasi-normal matrices:

If $A_0 + A_1 j$ and $B_0 + B_1 j$ are two quaternion quasi-normal matrices such that that $AB^C = BA^C$, that is $A_0 B_0^C - A_1 B_1^C j = B_0 A_0^C - B_1 A_1^C j$ then $A_0 + A_1 j$ and $B_0 + B_1 j$ can be simultaneously brought into the above quaternion normal form under the same $U_0 + U_1 j$ (with a generalization to a finite number) but not conversely; if $(A_0 + A_1 j)$ is quaternion quasi-normal $AA^C = A_0 A_0^C - A_1 A_1^C j$ is quaternion normal in the usual sense, but not conversely; and if $(A_0 + A_1 j)$ is quaternion quasi-normal and AA^C is real, there is a real orthogonal matrix which gives the above form.

Properties of double representation of quaternion quasi-normal matrices not obtained in this section but of subsequent use are the following:

a) $A = A_0 + A_1 j$ is both quaternion normal and quaternion quasi-normal matrices if and only if $A_0 + A_1 j = H_0 U_0 + H_1 U_1 j = U_0 H_0 + U_1 H_1 j = U_0 H_0^T + U_1 H_1^T j$ so $H = H_0 + H_1 j = H_0^T + H_1^T j = H_0^{CT} - H_1^{CT} j$ so that H is real.

b) If $A_0 + A_1 j = H_0 U_0 + H_1 U_1 j = U_0 H_0^T + U_1 H_1^T j$ is quaternion quasi-normal matrices, then $U_0 H_0 + U_1 H_1 j$ is quaternion quasi-normal matrices, if and only if $H_0 U_0^2 + H_1 U_1^2 j = U_0^2 H_0 + U_1^2 H_1 j$, that is if and only if $H_0 U_0^2 + H_1 U_1^2 j$ is quaternion normal.

For if $U_0 H_0 + U_1 H_1 j$ is quaternion quasi-normal matrices, $U_0 H_0 + U_1 H_1 j = H_0^T U_0 + H_1^T U_1 j$ so that $H_0 U_0^2 + H_1 U_1^2 j = U_0 H_0^T U_0 + U_1 H_1^T U_1 j = U_0^2 H_0 + U_1^2 H_1 j$ and if $H_0 U_0^2 + H_1 U_1^2 j = U_0^2 H_0 + U_1^2 H_1 j$, then $HUU = (H_0 + H_1 j)(U_0 + U_1 j)(U_0 + U_1 j) = U_0 U_0 H_0 + U_1 U_1 H_1 j = UUH$ and $H^T U = H_0^T U_0 + H_1^T U_1 j = U_0 H_0 + U_1 H_1 j = UH$

Theorem: 2

If $A_0 + A_1 j$ and $B_0 + B_1 j$ are quaternion quasi-normal matrices, then $A_0 B_0 + A_1 B_1 j$ is quaternion normal if and only if $A_0^{CT} A_0 B_0 - A_1^{CT} A_1 B_1 j = B_0 A_0 A_0^{CT} - B_1 A_1 A_1^{CT} j$ and $A_0 B_0 B_0^{CT} - A_1 B_1 B_1^{CT} j = B_0^{CT} B_0 A_0 - B_1^{CT} B_1 A_1 j$ (that is if and only if each is "quaternion normal relative to the other").

Proof:

If $A_0B_0 + A_1B_1j$ is quaternion normal, from the above, $D = D_0 + D_1j$, $D^{CT}DB_2 = D_0^{CT}D_0B_{0(2)} - D_1^{CT}D_1B_{1(2)}j = B_{0(2)}D_0D_0^{CT} - B_{1(2)}D_1D_1^{CT}j = B_2DD^{CT}$ so that $F_0^{CT}F_0B_{0(1)} - F_1^{CT}F_1B_{1(1)}j = B_{0(1)}F_0F_0^{CT} - B_{1(1)}F_1F_1^{CT}j$ or $A^{CT}AB = A_0^{CT}A_0B_0 - A_1^{CT}A_1B_1j = B_0A_0A_0^{CT} - B_1A_1A_1^{CT}j$.

Similarly, since $DB_2 = D_0B_{0(2)} + D_1B_{1(2)}j$ is quaternion normal, $DB_2B_2^{CT}D^C = D_0B_{0(2)}B_{0(2)}^{CT}D_0^C + D_1B_{1(2)}B_{1(2)}^{CT}D_1^Cj = B_{0(2)}^{CT}D_0^CD_0B_{0(2)} + B_{1(2)}^{CT}D_1^CD_1B_{1(2)}j = B_2^{CT}D^CDB_2$ so, $DB_2B_2^{CT} = D_0B_{0(2)}B_{0(2)}^{CT} - D_1B_{1(2)}B_{1(2)}^{CT}j = B_{0(2)}^{CT}B_{0(2)}D_0 - B_{1(2)}^{CT}B_{1(2)}D_1j$ or $FB_1B_1^{CT} = F_0B_{0(1)}B_{0(1)}^{CT} - F_1B_{1(1)}B_{1(1)}^{CT}j = B_{0(1)}^{CT}B_{0(1)}F_0 - B_{1(1)}^{CT}B_{1(1)}F_1j$ or $A_0B_0B_0^{CT} - A_1B_1B_1^{CT}j = B_0^{CT}B_0A_0 - B_1^{CT}B_1A_1j$. That is $ABB^{CT} = B^{CT}BA$. The converse is directly verifiable.

Double Representation Of Quaternion Quasi-Normal Products of Matrices:

It is possible if $A_0 + A_1j$ is quaternion normal and $B_0 + B_1j$ is quaternion quasi-normal that $A_0B_0 + A_1B_1j$ is quaternion quasi-normal.

For example

Any quaternion quasi-normal matrix $C = H_0U_0 + H_1U_1j = U_0H_0^T + U_1H_1^Tj$ is such a product with $A = A_0 + A_1j = H_0 + H_1j$ and $B_0 + B_1j = U_0 + U_1j$ or if $C = H_0U_0 + H_1U_1j = U_0H_0^T + U_1H_1^Tj$ and $A_0 + A_1j = H_0 + H_1j$ then $AC = (H_0 + H_1j)(H_0U_0 + H_1U_1j) = (H_0 + H_1j)(H_0U_0 + H_1U_1j) = (H_0U_0 + H_1U_1j)(H_0^T + H_1^Tj) = U_0(H_0^T)^2 + U_1(H_1^T)^2j$. Therefore AC is quaternion quasi-normal.

Theorem: 3

If $A = G_0W_0 + G_1W_1j = W_0G_0 + W_1G_1j$ is quaternion normal and $B = S_0V_0 + S_1V_1j = V_0S_0^T + V_1S_1^Tj$ is quaternion quasi-normal (where G_0, G_1, S_0, S_1 are hermitian and W_0, W_1, V_0, V_1 are unitary) then AB is quaternion quasi-normal if and only if $G_0S_0 + G_1S_1j = S_0G_0 + S_1G_1j$, $G_0V_0 + G_1V_1j = V_0G_0^T + V_1G_1^Tj$ and $W_0S_0 + W_1S_1j = S_0W_0 + S_1W_1j$.

Proof:

If the three relations hold, then $AB = G_0W_0S_0V_0 + G_1W_1S_1V_1j = G_0S_0W_0V_0 + G_1S_1W_1V_1j$ on one hand, and $AB = W_0G_0S_0V_0 + W_1G_1S_1V_1j = W_0S_0V_0G_0^T + W_1S_1V_1G_1^Tj = W_0V_0S_0^TG_0^T + W_1V_1S_1^TG_1^Tj = W_0V_0(G_0S_0)^T + W_1V_1(G_1S_1)^Tj$ is quaternion quasi-normal, since $G_0S_0 + G_1S_1j$ is hermitian and $W_0V_0 + W_1V_1j$ is unitary.

Conversely, let $A = U_0^{CT}D_0U_0 - U_1^{CT}D_1U_1j = G_0W_0 + G_1W_1j$ and $B = U_0^{CT}B_{0(1)}^TU_0^C + U_1^{CT}B_1^TU_1^Cj = (U_0^{CT}S_{0(1)}U_0 - U_1^{CT}S_{1(1)}U_1j)(U_0^{CT}V_{0(1)}U_0^C + U_1^{CT}V_{1(1)}U_1^Cj) = V_0S_0^T + V_1S_1^Tj$,where $S_{0(1)}$, $S_{1(1)}$ and $V_{0(1)}$, $V_{1(1)}$ are hermitian and unitary and direct sums conformable to $B_{0(1)}^T + B_{1(1)}^Tj$ and $D_0 + D_1j$.

A direct check shows that $G_0S_0 + G_1S_1j = S_0G_0 + S_1G_1j$ and $G_0V_0 + G_1V_1j = V_0G_0^T + V_1G_1^Tj$ also $W_0S_0 + W_1S_1j = U_0^{CT}D_{0(u)}K_{0(1)}U_0 - U_1^{CT}D_{1(u)}K_{1(1)}U_1j = U_0^{CT}K_{0(1)}D_{0(u)}U_0 - U_1^{CT}K_{1(1)}D_{1(u)}U_1j = S_0W_0 + S_1W_1j$. Since

$D_{0(u)}B_{0(1)}B_{0(1)}^{CT} - D_{1(u)}B_{1(1)}B_{1(1)}^{CT}j$ = $B_{0(1)}B_{0(1)}^{CT}D_{0(u)} - B_{1(1)}B_{1(1)}^{CT}D_{1(u)}j$ implies that $D_{0(u)}K_{0(1)} + D_{1(u)}K_{1(1)}j$ = $K_{0(1)}D_{0(u)} + K_{1(1)}D_{1(u)}j$.

Theorem: 4

If $A_0 + A_1 j$ is quaternion normal, $B_0 + B_1 j$ is quaternion quasi-normal, and $A_0B_0 + A_1B_1 j$ = $B_0A_0^T + B_1A_1^T j$, then $W_0A_0W_0^{CT} - W_1A_1W_1^{CT}j$ = $D_0 + D_1 j$ and $W_0B_0^TW_0 + W_1B_1^TW_1 j$ = $F_0 + F_1 j$, the quaternion normal form of Theorem 1, where $W_0 + W_1 j$ is a quaternion unitary matrix; also $A_0B_0 + A_1B_1 j$ is quaternion quasi-normal .

Proof:

Let $U_0A_0U_0^{CT} - U_1A_1U_1^{CT}j$ = $D_0 + D_1 j$ quaternion diagonal and $U_0B_0U_0^T + U_1B_1U_1^T j$ = $B_{0(2)} + B_{1(2)}j$ which is quaternion quasi-normal. Then $A_0B_0 + A_1B_1 j$ = $B_0A_0^T + B_1A_1^T j$, implies $D_0B_{0(2)} + D_1B_{1(2)}j$ = $U_0A_0U_0^{CT}U_0B_0U_0^T - U_1A_1U_1^{CT}U_1B_1U_1^T j$ = $U_0B_0U_0^TU_0^CA_0^TU_0^T - U_1B_1U_1^TU_1^CA_1^TU_1^T j$ = $B_{0(2)}D_0^T + B_{1(2)}D_1^T j$ = $B_{0(2)}D_0 + B_{1(2)}D_1 j$.

Let $D_0 + D_1 j$ = $C_1I_1 \oplus C_2I_2 \oplus .... \oplus C_mI_m$, where the C_p are quaternion and $C_p \neq C_q$ for $p \neq q$ and $C_{0(p)}, C_{1(p)}$, $B_{0(2)} + B_{1(2)}j$ = $C_1 \oplus C_2 \oplus .... \oplus C_m$. Let V_p be unitary such that $V_{0(p)}C_{0(p)}V_{0(p)}^T + V_{1(p)}C_{1(p)}V_{1(p)}^T j$ = $F_{0(p)} + F_{1(p)}j$ = the real quaternion normal form of Theorem 1, and let $V_0 + V_1 j$ = $V_1 \oplus V_2 \oplus .... \oplus V_m$. Then $V_0U_0A_0U_0^{CT}V_0^{CT} + V_1U_1A_1U_1^{CT}V_1^{CT}j$ = $D_0 + D_1 j$, $V_0U_0B_0U_0^TV_0^T + V_1U_1B_1U_1^TV_1^T j$ = $F_0 + F_1 j$ = a direct sum of the $F_{0(p)} + F_{1(p)}j$.

Also $A_0B_0 + A_1B_1 j$ = $B_0A_0^T + B_1A_1^T j$ implies that $B_0^TA_0^T + B_1^TA_1^T j$ = $A_0B_0^T + A_1B_1^T j$ and so $A_0B_0B_0^{CT}A_0^{CT} + A_1B_1B_1^{CT}A_1^{CT}j$ = $A_0B_0^TB_0^CA_0^{CT} + A_1B_1^TB_1^CA_1^{CT}j$ = $B_0^TA_0^TA_0^CB_0^C + B_1^TA_1^TA_1^CB_1^C j$ = $(A_0B_0)^T(A_0B_0)^C + (A_1B_1)^T(A_1B_1)^C j$ (The fact that $A_0 + A_1 j$ is quaternion normal is not used in the latter.)

It is also possible for the product of two quaternion normal matrices $A_0 + A_1 j$ and $B_0 + B_1 j$ to be quaternion quasi-normal. Let $Q_0 + Q_1 j$ = $H_0U_0 + H_1U_1 j$ = $U_0H_0^T + U_1H_1^T j$ is quaternion quasi-normal and if $A_0 + A_1 j$ = $U_0 + U_1 j$ and $B_0 + B_1 j$ = $H_0 + H_1 j$ this is so or if $S_0V_0 + S_1V_1 j$ = $V_0S_0^T + V_1S_1^T j$ is quaternion quasi-normal and if $A_0 + A_1 j$ = $U_0S_0 + U_1S_1 j$ = $S_0U_0 + S_1U_1 j$ is quaternion normal with S_0 and S_1 are hermitian and $V_0 + V_1 j$ and $U_0 + U_1 j$ are unitary, for $B_0 + B_1 j$ = $V_0 + V_1 j$, we have $A_0B_0 + A_1B_1 j$ = $(U_0S_0 + U_1S_1 j)\,(V_0 + V_1 j)$ = $(S_0 + S_1 j)\,(U_0V_0 + U_1V_1 j)$ = $U_0V_0S_0^T + U_1V_1S_1^T j$. It is a quaternion quasi-normal.`

But if in the first example $U_0^2H_0 + U_1^2H_1 j$ is not quaternion normal, then $H_0U_0 + H_1U_1 j$ is not quaternion quasi-normal. So that $B_0A_0 + B_1A_1 j$ is not necessarily quaternion quasi-normal though $A_0B_0 + A_1B_1 j$ is. When $A_0 + A_1 j$ alone is quaternion normal an analog of Theorem 2 can be obtained which states the following: If $A_0 + A_1 j$ is quaternion normal then $A_0B_0 + A_1B_1 j$ and $A_0B_0^T + A_1B_1^T j$ are quaternion quasi-normal if and only if $A_0B_0B_0^{CT} + A_1B_1B_1^{CT}j$ = $B_0^TB_0^CA_0 - B_1^TB_1^CA_1 j$ = $B_0B_0^{CT}A_0 - B_1B_1^{CT}A_1 j$ = $A_0B_0^TB_0^C - A_1B_1^TB_1^C j$ = $B_0^CA_0A_0^{CT} + B_1^CA_1A_1^{CT}$ = $A_0^TA_0^CB_0^C + A_1^TA_1^CB_1^C j$. (The proof is not included here because of its similarity to that above).

It is possible for the product of two quaternion quasi-normal matrices to be quaternion quasi-normal, but no such simple analogous necessary and sufficient conditions as exhibited above are available. This may be seen as follows. Two non-real quaternion commutative matrices $X_0 + X_1 j$ = $X_0^T + X_1^T j$ and $Y_0 + Y_1 j$ = $Y_0^T + Y_1^T j$ can form a quaternion quasi-normal (and non-real symmetric) matrix $X_0 Y_0 + X_1 Y_1 j$ (such that $Y_0 X_0 + Y_1 X_1 j$ is also quaternion quasi-normal) which need not be quaternion normal.

Then two symmetric matrices:

$$X_0 + X_1 j = \begin{bmatrix} i & 1+i \\ 1+i & -i \end{bmatrix}, \qquad Y_0 + Y_1 j = \begin{bmatrix} 1+2i & 3-4i \\ 3-4i & -(1+2i) \end{bmatrix}$$

Are such that $X_0 Y_0 + X_1 Y_1 j$ = $Z_0 + Z_1 j$ is real, quaternion normal and quaternion quasi-normal (and not symmetric). Finally, if $U_0 + U_1 j$ and $V_0 + V_1 j$ are two quaternion unitary matrices of the same order, they can be chosen so $U_0 V_0 + U_1 V_1 j$ is non-real quaternion, quaternion normal and quaternion quasi-normal.

If $A_0 + A_1 j$ = $(X_0 + X_1 j) + (S_0 + S_1 j) + (U_0 + U_1 j)$, $B_0 + B_1 j$ = $(Y_0 + Y_1 j) + (T_0 + T_1 j) + (V_0 + V_1 j)$. Then $A_0 B_0 + A_1 B_1 j$ = $(X_0 Y_0 + X_1 Y_1 j) + (S_0 T_0 + S_1 T_1 j) + (U_0 V_0 + U_1 V_1 j)$ where $A_0 + A_1 j$ and $B_0 + B_1 j$ are quaternion quasi-normal as in $A_0 B_0 + A_1 B_1 j$ (but not symmetric).

A simple inspection of these matrices shows that relations on the order of $(B_0^T B_0^C) A_0 - (B_1^T B_1^C) A_1 j$ = $A_0 (B_0 B_0^{CT}) - A_1 (B_1 B_1^{CT}) j$ = $(B_0 B_0^{CT}) A_0 - (B_1 B_1^{CT}) A_1 j$ and $(A_0^T A_0^C) B_0^C + (A_1^T A_1^C) B_1^C j$ = $(A_0 A_0^{CT}) B_0^C + (A_1 A_1^{CT}) B_1^C j$ = $B_0^C (A_0 A_0^{CT}) + B_1^C (A_1 A_1^{CT}) j$ do not necessarily hold; these are sufficient, however , to guarantee that $A_0 B_0 + A_1 B_1 j$ is quaternion quasi-normal (as direct verification from the definition will show).

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