Some Eigenvalue Inequalities For Positive Semidefinite Quaternion Hermitian Matrix Power Products

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Abstract:

In this paper, we study the eigenvalues of positive semidefinite Quaternion Hermitian matrix power products and obtain some inequalities, most of which are in terms of majorization. In particular, for $A, B \geq 0,\ \beta > \alpha > 0$, we prove $log\lambda^{\frac{1}{\alpha}}(A^{\alpha} * B^{\alpha}) < log\lambda^{\frac{1}{\beta}}(A^{\beta} * B^{\beta})$. The result is a generalization of some work of Marcus, Lieb, Thirring, Lecouteur, Bushell and Trustrum.

Keywords: Hermitian Matrices, Quaternion Matrices, Eigenvalue, Singular value, Positive semidefinite.

1. Introduction:

For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let $x \downarrow = (x_{[1]}, x_{[2]}, \dots, x_{[n]})$ denote the decreasing rearrangement of x, i.e., $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$, and $x \uparrow = (x_{(1)}, x_{(2)}, \dots, x_{(n)})$ denote the increasing rearrangement of x.

For $x, y \in \mathbb{R}^n$, x is said to be weakly majorized by y if

$\sum_{t=1}^{l} x_{[t]} \leq \sum_{t=1}^{l} y_{[t]}, l = 1,2,..,n.$ ... (1)

This relation is denoted by $x <_w y$. If, in addition to (1), equality occurs at $l = n$, then x is said to be majorized by y and we write $x < y$.

For $0 \leq x, y \in \mathbb{R}^n$, let us write $logx <_w logy$ to mean

$$\prod_{t=1}^{l} x_{[t]} \leq \prod_{t=1}^{l} y_{[t]}, l = 1,2, \dots, n$$

If equality occurs at $l = n$, we write it as $logx < logy$.

Let $A \in H_{n \times n}$ be quaternion matrix of the form $A_0 + A_1 j + A_2 k$, with $A_0, A_1, A_2 \in C_{n \times n}$. The eigenvalues and singular values of A are respectively denoted by $\lambda_1(A), \dots, \lambda_n(A)$ and $\sigma_1(A), \dots, \sigma_n(A)$ with

$|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$,

$\sigma_1(A) \geq \cdots \geq \sigma_n(A)$.

In particular, when A is positive semidefinite ($A \geq 0$), then

$\lambda_1(A) \geq \cdots \geq \lambda_n(A) \geq 0$ and $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A)) = \lambda(A) \downarrow$, and for $\alpha \in R$, $\lambda^\alpha(A) = \lambda(A^\alpha) = (\lambda_1^\alpha(A), \dots, \lambda_n^\alpha(A)) \downarrow$.

Theorem 1:

For the positive semidefinite quaternion square matrices $A, B \geq 0$ then $\log[\lambda(A) \circ \lambda(B)] \uparrow < \log[\lambda(AB)] < \log[\lambda(A) \circ \lambda(B)]$.

Proof:

Since $A, B \geq 0$, $A = A_0 + A_1 j + A_2 k$ and $B = B_0 + B_1 j + B_2 k$

$\lambda(A) = \lambda(A_0) + \lambda(A_1 j) + \lambda(A_2 k)$, $\lambda(B) = \lambda(B_0) + \lambda(B_1 j) + \lambda(B_2 k)$

$\lambda(AB) = \lambda(A_0 B_0) + \lambda(A_1 B_1 j) + \lambda(A_2 B_2 k)$ [Since $AB = A_0 B_0 + A_1 B_1 j + A_2 B_2 k$]

$\leq \lambda(A_0)\lambda(B_0) + \lambda(A_1)\lambda(B_1)j + \lambda(A_2)\lambda(B_2)k$ [Since $\lambda(AB) \leq \lambda(A)\lambda(B)$]

Here $\lambda(A) \uparrow = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))$ and $\lambda(B) \uparrow = (\lambda_1(B), \lambda_2(B), \dots, \lambda_n(B))$

$\lambda(A) \circ \lambda(B) = (\lambda_1(A)\lambda_1(B), \lambda_2(A)\lambda_2(B), \dots, \lambda_n(A)\lambda_n(B))$

$\prod_{t=1}^n \lambda_t(A) \circ \lambda_t(B) \uparrow = \lambda_{(1)}(A)\lambda_{(1)}(B)\lambda_{(2)}(A)\lambda_{(2)}(B) \dots \lambda_{(n)}(A)\lambda_{(n)}(B)$ … (2)

$\lambda(AB) = [\lambda_1(AB), \lambda_2(AB), \dots, \lambda_n(AB)]$

[Since $\lambda_t(AB) \leq \lambda_t(A)\lambda_t(B) \; \forall \; t = 1 \; to \; n$]

$\lambda(AB) \leq [\lambda_1(A)\lambda_1(B), \lambda_2(A)\lambda_2(B), \dots, \lambda_n(A)\lambda_n(B)]$

$\prod_{t=1}^n \lambda_{[t]}(AB) \leq \prod_{t=1}^n \lambda_t(A)\, \lambda_t(B)$ … (3)

From (2) and (3), $\prod_{t=1}^n \lambda_t(A)\, \lambda_t(B) \uparrow \leq \prod_{t=1}^n \lambda_{(t)}(AB)$

This implies $log\lambda(A) \circ \lambda(B) \uparrow < log\lambda(AB)$ … (4)

$\lambda(A) \circ \lambda(B) = (\lambda_1(A)\lambda_1(B), \lambda_2(A)\lambda_2(B), \dots, \lambda_n(A)\lambda_n(B))$

$log\lambda(A) \circ \lambda(B) = \prod_{t=1}^n \lambda_1(A)\, \lambda_1(B)\lambda_2(A)\lambda_2(B) \dots \lambda_n(A)\lambda_n(B) \; \forall \; t = 1 \; to \; n$ … (5)

From (3) and (5), $log\lambda(AB) < log\lambda(A) \circ \lambda(B)$

Thus $log[\lambda(A) \circ \lambda(B)] \uparrow < \log[\lambda(AB)] < log[\lambda(A) \circ \lambda(B)]$

The proof is completed.

Theorem 2:

For arbitrary quaternion herimitian matrices A and B,

$\log[\sigma(A) \circ \sigma(B)] \uparrow < \log[\sigma(AB)] < \log[\sigma(A) \circ \sigma(B)]$.

Proof:

$A = A_0 + A_1 j + A_2 k$ and $B = B_0 + B_1 j + B_2 k$

$\sigma(A) = \sigma(A_0) + \sigma(A_1 j) + \sigma(A_2 k)$ and $\sigma(B) = \sigma(B_0) + \sigma(B_1 j) + \sigma(B_2 k)$

$\sigma(AB) = \sigma(A_0B_0) + \sigma(A_1B_1j) + \sigma(A_2B_2k)$ [Since $AB = A_0B_0 + A_1B_1j + A_2B_2k$]

$\leq \sigma(A_0)\lambda(B_0) + \sigma(A_1)\lambda(B_1)j + \sigma(A_2)\lambda(B_2)k$

[Since $\sigma(AB) \leq \sigma(A)\sigma(B)$]

Here $\sigma(A)\uparrow = (\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A))$ and $\sigma(B)\uparrow = (\sigma_1(B), \sigma_2(B), \dots, \sigma_n(B))$

$\sigma(A) \circ \sigma(B) = (\sigma_1(A)\sigma_1(B), \sigma_2(A)\sigma_2(B), \dots, \sigma_n(A)\sigma_n(B))$

$\prod_{t=1}^{n} \sigma_t(A) \circ \sigma_t(B) \uparrow = \sigma_{(1)}(A)\sigma_{(1)}(B)\sigma_{(2)}(A)\sigma_{(2)}(B) \dots \sigma_{(n)}(A)\sigma_{(n)}(B)$ … (6)

$\sigma(AB) = \sigma_1(AB), \sigma_2(AB), \dots, \sigma_n(AB)$

$log\sigma(AB) = log\sigma_1(AB) + log\sigma_2(AB) + \cdots + log\sigma_n(AB) = \prod_{t=1}^{n} log\sigma_t(AB)$ … (7)

[Since $\sigma_t(AB) \leq \sigma_t(A)\sigma_t(B) \forall\, t = 1\ to\ n$]

$\prod_{t=1}^{n} \sigma_{[t]}(AB) \leq \prod_{t=1}^{n} \sigma_t(A)\, \sigma_t(B)$

This implies $log\sigma(AB) \leq log[\prod_{t=1}^{n} \sigma_t(AB)]$

(6) and (7) gives,

$\prod_{t=1}^{n} [\sigma_t(A)\, \sigma_t(B)] \uparrow \leq \prod_{t=1}^{n} \sigma_{(t)}(AB)$

This implies $\log[\sigma(A)\sigma(B)] \uparrow < log\sigma(AB)$ … (8)

$\sigma(A) \circ \sigma(B) = (\sigma_1(A)\sigma_1(B), \sigma_2(A)\sigma_2(B), \dots, \sigma_n(A)\sigma_n(B))$

$\log[\sigma(A) \circ \sigma(B)] = log[\sigma_1(A)\sigma_1(B)] + \cdots + \log[\sigma_n(A)\sigma_n(B)]$ … (9)

(7) and (9) gives,

$\prod_{t=1}^{n} log\sigma_t(AB) < \prod_{t=1}^{n} log\sigma_t(A)\sigma_t(B)$

$\log[\sigma(AB)] < \log[\sigma(A) \circ \sigma(B)]$

Thus $\log[\sigma(A) \circ \sigma(B)] \uparrow < \log[\sigma(AB)] < \log[\sigma(A) \circ \sigma(B)]$

The proof is completed.

Theorem 3:

For the positive semidefinite quaternion square matrices $A, B \geq 0, m \in N$. The following result is due to

$$\sum_{t=1}^{n} \lambda_t^m(A)\, \lambda_{n-t+1}^m(B) \leq trA^mB^m \leq \sum_{t=1}^{n} \lambda_t^m(A)\, \lambda_t^m(B)$$

Proof:

Since $A, B \geq 0$, $A = A_0 + A_1j + A_2k$ and $B = B_0 + B_1j + B_2k$

$\lambda(A) = \lambda(A_0 + A_1j + A_2k)$ and $\lambda(B) = \lambda(B_0 + B_1j + B_2k)$

Let us consider $\sum_{t=1}^{n} \lambda_t^m(A)\, \lambda_{n-t+1}^m(B)$ where

$\lambda_1^m(A) = \lambda_1^m(A_0 + A_1j + A_2k)$

$= \lambda_1(A_0 + A_1j + A_2k)^m$

$= \lambda_1({A_0}^m + {A_1}^m j + {A_2}^m k)$

$= \lambda_1({A_0}^m) + \lambda_1({A_1}^m j) + \lambda_1({A_2}^m k)$

$= \lambda_1^m(A_0) + \lambda_1^m(A_1 j) + \lambda_1^m(A_2 k)$

etc

$\lambda_n^m(A) = \lambda_n^m(A_0) + \lambda_n^m(A_1 j) + \lambda_n^m(A_2 k)$

Similarly,

$\lambda_1^m(B) = \lambda_1^m(B_0) + \lambda_1^m(B_1 j) + \lambda_1^m(B_2 k)$

etc

$\lambda_n^m(B) = \lambda_n^m(B_0) + \lambda_n^m(B_1 j) + \lambda_n^m(B_2 k)$

$\lambda_1^m(A)\lambda_n^m(B) = [\lambda_1^m(A_0) + \lambda_1^m(A_1 j) + \lambda_1^m(A_2 k)][\lambda_n^m(B_0) + \lambda_n^m(B_1 j) + \lambda_n^m(B_2 k)]$

$= \lambda_1^m(A_0)[\lambda_n^m(B)] + \lambda_1^m(A_1 j)[\lambda_n^m(B)] + \lambda_1^m(A_2 k)[\lambda_n^m(B)]$

$\sum_{t=1}^n \lambda_t^m(A)\, \lambda_{n-t+1}^m(B) = \lambda_1^m(A_0)\lambda_n^m(B_0) + \lambda_1^m(A_0)\lambda_n^m(B_1 j) + \lambda_1^m(A_0)\lambda_n^m(B_2 k) +$

$\lambda_1^m(A_1 j)\lambda_n^m(B_0) + \lambda_1^m(A_1 j)\lambda_n^m(B_1 j) + \lambda_1^m(A_1 j)\lambda_n^m(B_2 k) +$

$\lambda_1^m(A_2 k)\lambda_n^m(B_0) + \lambda_1^m(A_2 k)\lambda_n^m(B_1 j) +$

$\lambda_1^m(A_2 k)\lambda_n^m(B_2 k) + \cdots + \lambda_n^m(A_0)\lambda_1^m(B_0) +$

$\lambda_n^m(A_0)\lambda_1^m(B_1 j) + \lambda_n^m(A_0)\lambda_1^m(B_2 k) + \lambda_n^m(A_1 j)\lambda_1^m(B_0) +$

$\lambda_n^m(A_1 j)\lambda_1^m(B_1 j) + \lambda_n^m(A_1 j)\lambda_1^m(B_2 k) +$

$\lambda_n^m(A_2 k)\lambda_1^m(B_0) + \lambda_n^m(A_2 k)\lambda_1^m(B_1 j) + \lambda_n^m(A_2 k)\lambda_1^m(B_1 k)$

$= \sum_{t=1}^n \lambda_t^m(A_0)\, \lambda_{n-t+1}^m(B_0) + \sum_{t=1}^n \lambda_t^m(A_0)\, \lambda_{n-t+1}^m(B_1 j) +$

$\sum_{t=1}^n \lambda_t^m(A_0)\, \lambda_{n-t+1}^m(B_2 k) + \sum_{t=1}^n \lambda_t^m(A_1 j)\, \lambda_{n-t+1}^m(B_0) +$

$\sum_{t=1}^n \lambda_t^m(A_1 j)\, \lambda_{n-t+1}^m(B_1 j) + \sum_{t=1}^n \lambda_t^m(A_1 j)\, \lambda_{n-t+1}^m(B_2 k) +$

$\sum_{t=1}^n \lambda_t^m(A_2 k)\, \lambda_{n-t+1}^m(B_0) + \sum_{t=1}^n \lambda_t^m(A_2 k)\, \lambda_{n-t+1}^m(B_1 j) +$

$\sum_{t=1}^n \lambda_t^m(A_2 k)\, \lambda_{n-t+1}^m(B_2 k)$

$\le tr\ A_0^m B_0^m + tr\ A_0^m B_1^m j + tr\ A_0^m B_2^m k + tr\ A_1^m j B_0^m +$

$tr\ A_1^m B_1^m j + tr\ A_1^m B_2^m k + tr\ A_2^m k B_0^m +$ $tr\ A_2^m k B_1^m j +$

$tr\ A_2^m k B_2^m k$

Equating the corresponding elements on both sides,

$\sum_{t=1}^n \lambda_t^m(A_0)\, \lambda_{n-t+1}^m(B_0) \le tr\ A_0^m B_0^m$

$\sum_{t=1}^n \lambda_t^m(A_0)\, \lambda_{n-t+1}^m(B_1 j) \le tr\ A_0^m B_1^m j$

etc

$\sum_{t=1}^n \lambda_t^m(A_2 k)\, \lambda_{n-t+1}^m(B_2 k) \le tr\ A_2^m k B_2^m k$

Similarly we can prove that

$tr\,(A_0^m B_0^m) \leq \sum_{t=1}^{n} \lambda_t^m(A_0)\,\lambda_t^m(B_0)$

$$tr\,(A_0^m B_1^m j) \leq \sum_{t=1}^{n} \lambda_t^m(A_0)\,\lambda_t^m(B_1 j)$$

etc

$tr\,(A_2^m k B_2^m k) \leq \sum_{t=1}^{n} \lambda_t^m(A_2 k)\,\lambda_t^m(B_2 k)$

Thus $\sum_{t=1}^{n} \lambda_t^m(A)\,\lambda_{n-t+1}^m(B) \leq trA^m B^m \leq \sum_{t=1}^{n} \lambda_t^m(A)\,\lambda_t^m(B)$

The proof is completed.

Theorem 4:

For the positive semidefinite quaternion square matrices $A, B \geq 0, m \in N$.

The following result is due to

$$\sum_{t=1}^{n} \lambda_t^m(A)\,\lambda_{n-t+1}^m(B) \leq tr(AB)^m \leq \sum_{t=1}^{n} \lambda_t^m(A)\,\lambda_t^m(B)$$

Proof:

Since $A, B \geq 0$, $A = A_0 + A_1 j + A_2 k$ and $B = B_0 + B_1 j + B_2 k$

$\lambda(A) = \lambda(A_0 + A_1 j + A_2 k)$ and $\lambda(B) = \lambda(B_0 + B_1 j + B_2 k)$

Let us consider $\sum_{t=1}^{n} \lambda_t^m(A)\,\lambda_{n-t+1}^m(B)$ where

$\lambda_1^m(A) = \lambda_1^m(A_0) + \lambda_1^m(A_1 j) + \lambda_1^m(A_2 k)$

etc

$\lambda_n^m(A) = \lambda_n^m(A_0) + \lambda_n^m(A_1 j) + \lambda_n^m(A_2 k)$

Similarly $\lambda_1^m(B) = \lambda_1^m(B_0) + \lambda_1^m(B_1 j) + \lambda_1^m(B_2 k)$

etc

$\lambda_n^m(B) = \lambda_n^m(B_0) + \lambda_n^m(B_1 j) + \lambda_n^m(B_2 k)$

$$\begin{aligned}\lambda_1^m(A)\lambda_n^m(B) &= [\lambda_1^m(A_0) + \lambda_1^m(A_1 j) + \lambda_1^m(A_2 k)][\lambda_n^m(B_0) + \lambda_n^m(B_1 j) + \lambda_n^m(B_2 k)]\\ &\leq trA_o^m B_o^m + t_r A_o^m B_1^m j + trA_o^m B_2^m k + trA_1^m j B_o^m + trA_1^m j B_1^m j +\\ &\quad trA_1^m j B_2^m k + trA_2^m k B_o^m + trA_2^m k B_o^m + trA_2^m k B_1^m j + trA_2^m k B_2^m\end{aligned}$$

Equating the corresponding elements on both sides

$$\sum_{t=1}^{n} \lambda_t^m\,(A_0)\lambda_{n-t+1}^m(B_0) \leq trA_o^m B_o^m$$

$$\sum_{t=1}^{n} \lambda_t^m\,(A_0)\lambda_{n-t+1}^m(B_1 j) \leq trA_o^m B_1^m j$$

etc

$$\sum_{t=1}^{n} \lambda_t^m (A_2k)\lambda_{n-t+1}^m(B_2k) \leq trA_2^m kB_o^m k$$

Similarly we can prove that

$$tr(A_o^m B_o^m) \leq \sum_{t=1}^{n} \lambda_t^m (A_0)\lambda_t^m(B_0)$$

$$tr(A_o^m B_1^m j) \leq \sum_{t=1}^{n} \lambda_t^m (A_0)\lambda_t^m(B_1 j)$$

etc

$$tr(A_2^m kB_2^m k) \leq \sum_{t=1}^{n} \lambda_t^m (A_2k)\lambda_t^m(B_2k)$$

Here $\sum_{t=1}^{n} \lambda_t^m (A)\lambda_{n-t+1}^m(B_0) \leq trA^m B^m$

and $trA^m B^m \leq tr(AB)^m$

Thus $\sum_{t=1}^{n} \lambda_t^m (A)\lambda_{n-t+1}^m(B) \leq tr(AB)^m \leq \sum_{t=1}^{n} \lambda_t^m (A)\lambda_t^m(B_0)$

The Proof is completed.

Theorem 5 :

For the positive semi definite quaternion hermitian square matrices $A, B \geq 0, m \in N$. The following results is due to $tr(AB)^m \leq trA^m B^m$

Proof:

Since $A = A_0 + A_1 j + A_2 k$ and $B = B_0 + B_1 j + B_2 k$, $tr(AB)^m \leq tr\ AB$

$AB = A_0B_0 + A_1B_1 j + A_2B_2 k$

$(AB)^m = A_0B_0 \dots m\ times + A_1B_1 j \dots m\ times + A_2B_2 k \dots m\ times$

$tr(AB)^m = tr[A_0B_0 \dots + A_1B_1 j \dots + A_2B_2 k \dots]$

$= tr(A_0B_0)^m + tr(A_1B_1)^m j + tr(A_2B_2)^m k$

$\leq trA_0^m B_0^m + trA_1^m B_1^m j + trA_2^m B_2^m k$

$= tr[A_0^m B_0^m + A_1^m B_1^m j + A_2^m B_2^m k]$

$= trA^m B^m$

Thus $tr(AB)^m \leq trA^m B^m$

The Proof is completed.

Lemma 1:

(i) For n x n positive semi definite quaternion hermitian matrices A,B

$\lambda_1(AB) \leq \lambda_1(A)\lambda_1(B)$

$\lambda_n(AB) \leq \lambda_n(A)\lambda_n(B)$

Proof:

By the triple representation of quaternion matrices

$A = A_0 + A_1j + A_2k,\ B = B_0 + B_1j + B_2k$ and $AB = A_0B_0 + A_1B_1j + A_2B_2k$

$\lambda_1(AB) = \lambda_1(A_0B_0 + A_1B_1j + A_2B_2k)$

$= \lambda_1(A_0B_0) + \lambda_1(A_1B_1j) + \lambda_1(A_2B_2k)$

$\leq \lambda_1(A_0)\lambda_1(B_0) + \lambda_1(A_1)\lambda_1(B_1j) + \lambda_1(A_2)\lambda_1(B_2k)$ [Lemma 1 (i)]

$= \lambda_1(A)\lambda_1(B)$

$\lambda_1(AB \leq \lambda_1(A)\lambda_1(B)$

$\lambda_n(AB) = \lambda_n(A_0B_0) + \lambda_n(A_1B_1j) + \lambda_n(A_2B_2k)$

$\geq \lambda_n(A_0)\lambda_n(B_0) + \lambda_n(A_1)\lambda_n(B_1j) + \lambda_n(A_2)\lambda_n(B_2k)$ [Lemma 1 (i)]

$= \lambda_n(A)\lambda_n(B)$

Since $\lambda_n(AB) \geq \lambda_n(A)\lambda_n(B)$

The proof is completed.

(ii) For arbitrary n x n quaternion hermitian matrices X, Y

$|\lambda_1(XY)| \leq \sigma_1(X)\sigma_1(Y)$

$|\lambda_n(XY)| \geq \sigma_n(X)\sigma_n(Y)$

Proof:

$X = X_0 + X_1j + X_2k,\ Y = Y_0 + Y_1j + Y_2k$

$XY = X_0Y_0 + X_1Y_1j + X_2Y_2k$

$|\lambda_1(XY)| = |\lambda_1(X_0Y_0) + \lambda_1(X_1Y_1j) + \lambda_1(X_2Y_2k)|$

$\leq |\lambda_1(X_0Y_0)| + |\lambda_1(X_1Y_1j))| + |\lambda_1(X_2Y_2k)|$

$\leq \sigma_1(X_0Y_0) + \sigma_1(X_1Y_1)j + \sigma_1(X_2Y_2)k$

$\leq \sigma_1(X_0)\sigma_1(Y_0) + \sigma_1(X_1)\sigma_1(Y_1)j + \sigma_1(X_2)\sigma_1(Y_2)k$ [Lemma 1 (ii)]

$\leq \sigma_1(X)\sigma_1(Y)$

$|\lambda_1(XY)| \leq \sigma_1(X)\sigma_1(Y)$

$|\lambda_n(XY)| = |\lambda_n(X_0Y_0) + \lambda_n(X_1Y_1j) + \lambda_n(X_2Y_2k)|$

$\geq |\lambda_n(X_0Y_0)| + |\lambda_n(X_1Y_1j))| + |\lambda_n(X_2Y_2k)|$ [Lemma 1 (ii)]

$\geq \sigma_n(X_0Y_0) + \sigma_n(X_1Y_1)j + \sigma_n(X_2Y_2)k$

$\geq \sigma_n(X_0)\sigma_n(Y_0) + \sigma_n(X_1)\sigma_n(Y_1)j + \sigma_n(X_2)\sigma_n(Y_2)k$

$\geq \sigma_n(X)\sigma_n(Y)$

Hence $|\lambda_n(XY)| \geq \sigma_n(X)\sigma_n(Y)$

The proof is completed.

Theorem 6

For n x n positive semidefinite quaternion hermitian matrices A, B and $m \in N$

$$\lambda_1(AB) \leq \lambda_1^{\frac{1}{m}}(A^m B^m) \leq \lambda_1^{\frac{1}{m+1}}(A^{m+1}B^{m+1}) \leq \lambda_1(A)\lambda_1(B)$$

$$\lambda_n(AB) \geq \lambda_n^{\frac{1}{m}}(A^m B^m) \geq \lambda_n^{\frac{1}{m+1}}(A^{m+1}B^{m+1}) \geq \lambda_n(A)\lambda_n(B)$$

Proof:

$A = A_0 + A_1 j + A_2 k,\ B = B_0 + B_1 j + B_2 k$

$AB = A_0 B_0 + A_1 B_1 j + A_2 B_2 k$

$\lambda_1(AB) = \lambda_1(A_0 B_0 + A_1 B_1 j + A_2 B_2 k)$

Use induction on m to prove

$$\lambda_1(AB) \leq \lambda_1^{\frac{1}{m}}(A_0^m B_0^m + A_1^m B_1^m j + A_2^m B_2^m k)$$

$$\leq \lambda_1^{\frac{1}{m}}(A_0^m B_0^m) + \lambda_1^{\frac{1}{m}}(A_1^m)\lambda_1^{\frac{1}{m}}(B_1^m j) + \lambda_1^{\frac{1}{m}}(A_2^m)\lambda_1^{\frac{1}{m}}(B_2^m k)$$

$$\leq \lambda_1^{\frac{1}{m}}(A^m B^m) \qquad \text{[Theorem 6]}$$

$$\lambda_1(AB) \leq \lambda_1^{\frac{1}{m}}(A^m B^m) \qquad \ldots (1)$$

$$\lambda_1^{\frac{1}{m}}(A^m B^m) \leq \lambda_1^{\frac{1}{m+1}}(A_0^{m+1}B_0^{m+1}) + \lambda_1^{\frac{1}{m+1}}(A_1^{m+1}B_1^{m+1}j) + \lambda_1^{\frac{1}{m+1}}(A_2^{m+1}B_2^{m+1}k)$$

$$\leq \lambda_1^{\frac{1}{m+1}}(A^{m+1}B^{m+1})$$

$$\lambda_1^{\frac{1}{m}}(A^m B^m) \leq \lambda_1^{\frac{1}{m+1}}(A^{m+1}B^{m+1}) \qquad \ldots (2)$$

$$\lambda_1^{\frac{1}{m+1}}(A^{m+1}B^{m+1}) \leq \lambda_1(A_0 B_0 + A_1 B_1 j + A_2 B_2 k)$$

$$\leq \lambda_1(A_0 B_0) + \lambda_1(A_1 B_1 j) + \lambda_1(A_2 B_2 k)$$

$$\leq \lambda_1(A_0)\lambda_1(B_0) + \lambda_1(A_1)\lambda_1(B_1 j) + \lambda_1(A_2)\lambda_1(A_2 k)$$

$$\leq \lambda_1(A)\lambda_1(B)$$

$$\lambda_1^{\frac{1}{m+1}}(A^{m+1}B^{m+1}) \leq \lambda_1(A)\lambda_1(B) \qquad \ldots (3)$$

Combining (1), (2) and (3), we get,

$$\lambda_1(AB) \leq \lambda_1^{\frac{1}{m}}(A^m B^m) \leq \lambda_1^{\frac{1}{m+1}}(A^{m+1}B^{m+1}) \leq \lambda_1(A)\lambda_1(B)$$

By using lemma (1),

$\lambda_n(AB) \geq \lambda_n^{\frac{1}{m}}(A^m B^m) \geq \lambda_n^{\frac{1}{m+1}}(A^{m+1}B^{m+1}) \geq \lambda_n(A)\lambda_n(B)$ [Theorem 6]

The proof is completed.

Corollary: 1

For n x n normal quaternion hermitian matrices A, B and $m \in N$,

(i) $\sigma_1(AB) \leq \sigma_1^{\frac{1}{m}}(A^m B^m) \leq \sigma_1^{\frac{1}{(m+1)}}(A^{m+1}B^{m+1}) \leq \sigma_1(A)\sigma_1(B)$

(ii) $\sigma_n(AB) \geq \sigma_n^{\frac{1}{m}}(A^m B^m) \geq \sigma_n^{\frac{1}{(m+1)}}(A^{m+1}B^{m+1}) \geq \sigma_n(A)\sigma_n(B)$

Proof:

(i) Notice that $AA^* = A^*A, BB^* = B^*B$, we have

$AA^* = A_0A_0^* - A_1A_1^*j - A_2A_2^*k$,$A^*A = A_0^*A_0 - A_1^*A_1j - A_2^*A_2k$ [Since $AA^* = A^*A$]

$$\sigma_1^{\frac{1}{m}}(A^m B^m) = \lambda_1^{\frac{1}{2m}}(A^m B^m (B^*)^m (A^*)^m)$$

$$= \left[\lambda_1^{\frac{1}{m}}(AA^*)^m(BB^*)^m\right]^{\frac{1}{2}}$$

$$= \left[\lambda_1^{\frac{1}{m}}[(A_0^m(A_0^*)^m - A_1^m(A_1^*)^m j - A_2^m(A_2^*)^m k)(B_0^m(B_0^*)^m - B_1^m(B_1^*)^m j - B_2^m(B_2^*)^m k)]\right]^{\frac{1}{2}}$$

$$= \left[\lambda_1^{\frac{1}{m}}[A_0^m(A_0^*)^m B_0^m(B_0^*)^m] + \lambda_1^{\frac{1}{m}}[A_1^m(A_1^*)^m B_1^m(B_1^*)^m j] + \lambda_1^{\frac{1}{m}}[A_2^m(A_2^*)^m B_2^m(B_2^*)^m k]\right]^{\frac{1}{2}}$$

$$\leq \left[\lambda_1^{\frac{1}{m}}(A_0A_0^*)^m(B_0B_0^*)^m\right]^{\frac{1}{2}} + \left[\lambda_1^{\frac{1}{m}}(A_1A_1^*)^m(B_1B_1^*)^m j\right]^{\frac{1}{2}} + \left[\lambda_1^{\frac{1}{m}}(A_2A_2^*)^m(B_2B_2^*)^m k\right]^{\frac{1}{2}}$$

$$\leq \left[\lambda_1^{\frac{1}{m+1}}[(A_0A_0^*)^{m+1}(B_0B_0^*)^{m+1}]\right]^{\frac{1}{2}} + \left[\lambda_1^{\frac{1}{m+1}}[(A_1A_1^*)^{m+1}(B_1B_1^*)^{m+1}j]\right]^{\frac{1}{2}} + \left[\lambda_1^{\frac{1}{m+1}}[(A_2A_2^*)^{m+1}(B_2B_2^*)^{m+1}k]\right]^{\frac{1}{2}}$$

$$= \sigma_1^{\frac{1}{(m+1)}}(A_0^{m+1}B_0^{m+1}) + \sigma_1^{\frac{1}{(m+1)}}(A_1^{m+1}B_1^{m+1})j + \sigma_1^{\frac{1}{(m+1)}}(A_2^{m+1}B_2^{m+1})k$$

$$= \sigma_1^{\frac{1}{(m+1)}}(A^{m+1}B^{m+1})$$

$$\leq \sigma_1(A)\sigma_1(B)$$

Hence $\sigma_1(AB) \leq \sigma_1^{\frac{1}{m}}(A^mB^m) \leq \sigma_1^{\frac{1}{(m+1)}}(A^{m+1}B^{m+1}) \leq \sigma_1(A)\sigma_1(B)$

The proof of first part is completed.

(ii) $\sigma_n^{\frac{1}{m}}(A^mB^m) \quad = \lambda_n^{\frac{1}{2m}}(A^mB^m(B^*)^m(A^*)^m)$

$$= \left[\lambda_n^{\frac{1}{m}}(AA^*)^m(BB^*)^m\right]^{\frac{1}{2}}$$

$$= \left[\lambda_n^{\frac{1}{m}}[(A_0^m(A_0^*)^m - A_1^m(A_1^*)^mj - A_2^m(A_2^*)^mk)(B_0^m(B_0^*)^m - B_1^m(B_1^*)^mj - B_2^m(B_2^*)^mk)]\right]^{\frac{1}{2}}$$

$$= \left[\lambda_n^{\frac{1}{m}}[A_0^m(A_0^*)^mB_0^m(B_0^*)^m] + \lambda_n^{\frac{1}{m}}[A_1^m(A_1^*)^mB_1^m(B_1^*)^mj] + \lambda_n^{\frac{1}{m}}[A_2^m(A_2^*)^mB_2^m(B_2^*)^mk]\right]^{\frac{1}{2}}$$

$$\geq \left[\lambda_n^{\frac{1}{m}}(A_0A_0^*)^m(B_0B_0^*)^m\right]^{\frac{1}{2}} + \left[\lambda_n^{\frac{1}{m}}(A_1A_1^*)^m(B_1B_1^*)^mj\right]^{\frac{1}{2}} + \left[\lambda_n^{\frac{1}{m}}(A_2A_2^*)^m(B_2B_2^*)^mk\right]^{\frac{1}{2}}$$

$$\geq \left[\lambda_n^{\frac{1}{m+1}}[(A_0A_0^*)^{m+1}(B_0B_0^*)^{m+1}]\right]^{\frac{1}{2}} + \left[\lambda_n^{\frac{1}{m+1}}[(A_1A_1^*)^{m+1}(B_1B_1^*)^{m+1}j]\right]^{\frac{1}{2}} + \left[\lambda_n^{\frac{1}{m+1}}[(A_2A_2^*)^{m+1}(B_2B_2^*)^{m+1}k]\right]^{\frac{1}{2}}$$

$$= \sigma_n^{\frac{1}{(m+1)}}(A_0^{m+1}B_0^{m+1}) + \sigma_n^{\frac{1}{(m+1)}}(A_1^{m+1}B_1^{m+1})j + \sigma_n^{\frac{1}{(m+1)}}(A_2^{m+1}B_2^{m+1})k$$

$$= \sigma_n^{\frac{1}{(m+1)}}(A^{m+1}B^{m+1})$$

$$\geq \sigma_n(A)\sigma_n(B)$$

Hence $\sigma_n(AB) \geq \sigma_n^{\frac{1}{m}}(A^mB^m) \geq \sigma_n^{\frac{1}{(m+1)}}(A^{m+1}B^{m+1}) \geq \sigma_n(A)\sigma_n(B)$

The proof of second part is completed.

Lemma 2:

Let $X^{(l)}$ be the l^{th} compound of an n x n quaternion hermitian matrix X $(1 \leq l \leq n)$. Then (i) $(XY)^{(l)} = X^{(l)}Y^{(l)}$ and (ii) $\lambda_1(X^{(l)}) = \prod_{t=1}^{l}\lambda_t(X)$

Proof:

(i) $X = X_0 + X_1j + X_2k$ and $Y = Y_0 + Y_1j + Y_2k$

$XY = X_0Y_0 + X_1Y_1j + X_2Y_2k$

$X^l = X_0^l + X_1^lj + X_2^lk$, $X^{(l)} = X_0^{(l)} + X_1^{(l)}j + X_2^{(l)}k$

l^{th} compound of X is $X^{(l)}$ in which $X_0^{(l)}, X_1^{(l)}$ and $X_2^{(l)}$ are the l^{th} compound of X_0^l, X_1^l and X_2^l respectively.

$(X_0Y_0)^{(l)} = X_0^{(l)}Y_0^{(l)}$, $(X_1Y_1)^{(l)} = X_1^{(l)}Y_1^{(l)}$, $(X_2Y_2)^{(l)} = X_2^{(l)}Y_2^{(l)}$

$(X_0Y_0)^{(l)} + (X_1Y_1)^{(l)}j + (X_2Y_2)^{(l)}k = X_0^{(l)}Y_0^{(l)} + X_1^{(l)}Y_1^{(l)}j + X_2^{(l)}Y_2^{(l)}k$

Therefore $(XY)^{(l)} = X^{(l)}Y^{(l)}$

(ii) $\lambda_1(X^{(l)}) = \lambda_1\left(X_0^{(l)} + X_1^{(l)}j + X_2^{(l)}k\right)$

$= \lambda_1\left(X_0^{(l)}\right) + \lambda_1\left(X_1^{(l)}\right)j + \lambda_1\left(X_2^{(l)}\right)k$

$\lambda_1(X^{(l)}) = \prod_{t=1}^{l}\lambda_t(X_0) + \prod_{t=1}^{l}\lambda_t(X_1)j + \prod_{t=1}^{l}\lambda_t(X_2)\,k$

$= \prod_{t=1}^{l}[\lambda_t(X_0) + \lambda_t(X_1)j + \lambda_t(X_2)k]$

Therefore $\lambda_1(X)^{(l)} = \prod_{t=1}^{l}\lambda_t(X)$

The proof is completed.

Result : 1

For a positive definite quaternion hermitian matrix $A(A > 0), \alpha \in \mathbb{R}$,

$$(A^\alpha)^l = \left(A^{(l)}\right)^\alpha$$

Proof:

$A = A_0 + A_1 j + A_2 k$, $A^\alpha = {A_0}^\alpha + {A_1}^\alpha j + {A_2}^\alpha k$

$(A^\alpha)^l = ({A_0}^\alpha + {A_1}^\alpha j + {A_2}^\alpha k)^l$

$= ({A_0}^\alpha)^l + ({A_1}^\alpha)^l j + ({A_2}^\alpha)^l k$

$= \left({A_0}^l\right)^\alpha + \left({A_1}^l\right)^\alpha j + \left({A_2}^l\right)^\alpha k$

$(A^\alpha)^l = \left(A^{(l)}\right)^\alpha$

The proof is completed.

Theorem 7:

For n x n positive semidefinite quaternion hermitian matrices A, B and $m \in N$, $\log[\lambda(AB)] < log\left[\lambda^{\frac{1}{m}}(A^m B^m)\right] < log\left[\lambda^{\frac{1}{m+1}}(A^{m+1} B^{m+1})\right] < log[\lambda(A) \circ \lambda(B)]$

Proof:

We prove $log\lambda^{\frac{1}{m}}(A^m B^m) < log\lambda^{\frac{1}{m+1}}(A_0^{m+1} B_0^{m+1} + A_1^{m+1} B_1^{m+1} j + A_2^{m+1} B_2^{m+1} k)$;

the remaining part can be obtained similarly.

From lemma 2,

$$(A^m B^m)^{(l)} = (A_0^m + A_1^m j + A_2^m k)^{(l)} (B_0^m + B_1^m j + B_2^m k)^{(l)}$$

$$= \left(A_0^{(l)} + A_1^{(l)} j + A_2^{(l)} k\right)^m \left(B_0^{(l)} + B_1^{(l)} j + B_2^{(l)} k\right)^m$$

and $A^{(l)}, B^{(l)}$ are positive semidefinite quaternion hermitian matrix. Using Theorem 6,

$$\prod_{t=1}^{l} \lambda_t^{\frac{1}{m}}(A^m B^m) = \lambda_1^{\frac{1}{m}}\left(\left(A^{(l)}\right)^m \left(B^{(l)}\right)^m\right)$$

$$= \lambda_1^{\frac{1}{m}}\left(\left(A_0^{(l)} + A_1^{(l)} j + A_2^{(l)} k\right)^m \left(B_0^{(l)} + B_1^{(l)} j + B_2^{(l)} k\right)^m\right)$$

$$\leq \lambda_1^{\frac{1}{m+1}}\left(\left(A_0^{(l)} + A_1^{(l)} j + A_2^{(l)} k\right)^{m+1} \left(B_0^{(l)} + B_1^{(l)} j + B_2^{(l)} k\right)^{m+1}\right)$$

$$= \prod_{t=1}^{l} \lambda_t^{\frac{1}{m+1}}(A_0^{m+1} B_0^{m+1} + A_1^{m+1} B_1^{m+1} j + A_2^{m+1} B_2^{m+1} k),$$

$$l = 1,2, \dots, n$$

At $l = n$

$\prod_{t=1}^{l} \lambda_t^{\frac{1}{m}}(A^m B^m) = \prod_{t=1}^{l} \lambda_t^{\frac{1}{m+1}}(A_0^{m+1}B_0^{m+1} + A_1^{m+1}B_1^{m+1}j + A_2^{m+1}B_2^{m+1}k)$

So, $\log\left[\lambda^{\frac{1}{m}}(A^m B^m)\right] < log\left[\lambda^{\frac{1}{m+1}}(A^{m+1}B^{m+1})\right]$

The proof is completed.

Theorem 8:

For n x n positive semidefinite quaternion hermitian matrices A, B and $m \in N$

$$log[\lambda(A) \circ \lambda(B)] \uparrow< log\left[\lambda^{m+1}\left(A^{\frac{1}{(m+1)}}B^{\frac{1}{(m+1)}}\right)\right] < log\left[\lambda^m\left(A^{\frac{1}{m}}B^{\frac{1}{m}}\right)\right] < log[\lambda(AB)].$$

Proof:

Since $log\lambda\left(A^{\frac{1}{(m+1)}}\right) \circ \lambda\left(B^{\frac{1}{(m+1)}}\right) \uparrow< log\lambda\left(A^{\frac{1}{(m+1)}}B^{\frac{1}{(m+1)}}\right)$

From theorem 1, we have

$$log\lambda^{m+1}\left(A^{\frac{1}{(m+1)}}\right) \circ \lambda^{m+1}\left(B^{\frac{1}{(m+1)}}\right) \uparrow$$
$$< log\lambda^{m+1}\left(A_0^{\frac{1}{(m+1)}}B_0^{\frac{1}{(m+1)}} + A_1^{\frac{1}{(m+1)}}B_1^{\frac{1}{(m+1)}}j + A_2^{\frac{1}{(m+1)}}B_2^{\frac{1}{(m+1)}}k\right)$$

i.e., $log\lambda(A) \circ \lambda(B) \uparrow < log\lambda^{m+1}\left(A_0^{\frac{1}{(m+1)}}B_0^{\frac{1}{(m+1)}} + A_1^{\frac{1}{(m+1)}}B_1^{\frac{1}{(m+1)}}j + A_2^{\frac{1}{(m+1)}}B_2^{\frac{1}{(m+1)}}k\right)$

Also, for $G, H \geq 0$, Take $A = G^{\frac{1}{m(m+1)}}, B = H^{\frac{1}{m(m+1)}}$ is theorem 6,

We have $\lambda_1^{m+1}\left(G^{\frac{1}{(m+1)}}H^{\frac{1}{(m+1)}}\right) = \left[\lambda_1^{\frac{1}{m}}(A_0^m B_0^m + A_1^m B_1^m j + A_2^m B_2^m k)\right]^{m(m+1)}$

$$\leq \left[\lambda_1^{\frac{1}{m+1}}(A_0^{m+1}B_0^{m+1} + A_1^{m+1}B_1^{m+1}j + A_2^{m+1}B_2^{m+1}k)\right]^{m(m+1)}$$

$$\lambda_1^{m+1}\left(G^{\frac{1}{(m+1)}}H^{\frac{1}{(m+1)}}\right) = \lambda_1^m\left(G^{\frac{1}{m}}H^{\frac{1}{m}}\right)$$

Thus, $\lambda_1^{m+1}\left(G^{\frac{1}{(m+1)}}H^{\frac{1}{(m+1)}}\right) \leq \lambda_1^m\left(A_0^{\frac{1}{m}}B_0^{\frac{1}{m}} + A_1^{\frac{1}{m}}B_1^{\frac{1}{m}}j + A_2^{\frac{1}{m}}B_2^{\frac{1}{m}}k\right)$

$$\leq \lambda_1(A_0B_0 + A_1B_1j + A_2B_2k)$$

and using methods similar to those in the proof of theorem 7, it follows that

$$log\left[\lambda^{m+1}\left(A^{\frac{1}{(m+1)}}B^{\frac{1}{(m+1)}}\right)\right] < log\left[\lambda^m\left(A^{\frac{1}{m}}B^{\frac{1}{m}}\right)\right] < log[\lambda(AB)]$$

The proof is completed.

Corollary 2:

For $A, B \geq 0$ and $m \in N$,

$\log[\lambda^m(A) \circ \lambda^m(B)] \uparrow < log\ [\lambda^m(AB)] < \log[\lambda\ (A^m B^m)] < \log[\lambda^m(A) \circ \lambda^m(B)]$

Proof:

Since $A, B \geq 0$ and $m\ \epsilon\ N$

$A = A_0 + A_1 j + A_2 k$ and $B = B_0 + B_1 j + B_2 k$

$\lambda(A) = \lambda(A_0) + \lambda(A_1 j) + \lambda(A_2 k),\ \lambda(B) = \lambda(B_0) + \lambda(B_1 j) + \lambda(B_2 k)$

$\lambda^m(A) = \lambda^m(A_0) + \lambda^m(A_1 j) + \lambda^m(A_2 k)\ , \lambda^m(B) = \lambda^m(B_0) + \lambda^m(B_1 j) + \lambda^m(B_2 k)$

$\lambda(AB) = \lambda(A_0 B_0) + \lambda(A_1 B_1 j) + \lambda(A_2 B_2 k)$ [Since $AB = A_0 B_0 + A_1 B_1 j + A_2 B_2 k$]

$\lambda(AB) \leq \lambda(A_0)\lambda(B_0) + \lambda(A_1)\lambda(B_1 j) + \lambda(A_2)\lambda(B_2 k)$ [Since $\lambda(AB) \leq \lambda(A)\lambda(B)$]

Therefore, $\lambda^m(AB) \leq \lambda^m(A_0)\lambda^m(B_0) + \lambda^m(A_1)\lambda^m(B_1 j) + \lambda^m(A_2)\lambda^m(B_2 k)$

Here $\lambda(A) \uparrow = (\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))$ and $\lambda(B) \uparrow = (\lambda_1(B), \lambda_2(B), \dots, \lambda_n(B))$

$\lambda^m(A) \uparrow = (\lambda_1^m(A), \lambda_2^m(A), \dots, \lambda_n^m(A))$ and $\lambda^m(B) \uparrow = (\lambda_1^m(B), \lambda_2^m(B), \dots, \lambda_n^m(B))$

Therefore $\lambda^m(A) \circ \lambda^m(B) = (\lambda_1^m(A)\lambda_1^m(B), \lambda_2^m(A)\lambda_2^m(B), \dots, \lambda_n^m(A)\lambda_n^m(B))$

$\prod_{t=1}^n [\lambda_t^m(A) \circ \lambda_t^m(B)] \uparrow = \lambda_{(1)}^m(A)\ \lambda_{(1)}^m(B)\ \lambda_{(2)}^m(A)\ \lambda_{(2)}^m(B) \dots \lambda_{(n)}^m(A)\ \lambda_{(n)}^m(B)$ … (1)

$\lambda^m(AB) = [\lambda_1^m(AB), \lambda_2^m(AB), \dots, \lambda_n^m(AB)] \leq$

$[\lambda_1^m(A)\ \lambda_1^m(B), \lambda_2^m(A)\ \lambda_2^m(B), \dots, \lambda_n^m(A)\ \lambda_n^m(B)]$

[$\because\ \lambda_t^m(AB) \leq \lambda_t^m(A)\ \lambda_t^m(B) \forall\ t = 1\ to\ n.$]

$\prod_{t=1}^n \lambda_{[t]}^m(AB) \leq \prod_{t=1}^n \lambda_t^m(A)\lambda_t^m(B)$ … (2)

From (1) and (2),

$\prod_{t=1}^n \lambda_t^m(A)\lambda_t^m(B) \uparrow \leq \prod_{t=1}^n \lambda_{(t)}^m(AB)$

$\Rightarrow \log[\lambda^m(A) \circ \lambda^m(B)] \uparrow < \log[\lambda^m(AB)]$ … (3)

$\log \lambda^m(AB) < \log \lambda(AB)^m$

$\log \lambda^m(AB) < \log \lambda(A^m B^m)$ … (4)

$\lambda^m(A)\ \circ\ \lambda^m(B) = (\lambda_1^m(A)\lambda_1^m(B), \lambda_2^m(A)\lambda_2^m(B), \dots, \lambda_n^m(A)\lambda_n^m(B))$

$\log \lambda^m(A)\ \circ\ \lambda^m(B) = \prod_{t=1}^n \lambda_1^m(A)\lambda_1^m(B)\ \lambda_2^m(A)\lambda_2^m(B), \dots, \lambda_n^m(A)\lambda_n^m(B)$

$\forall\ t = 1\ to\ n$ … (5)

$\log \lambda\ (A^m B^m) < \log \lambda^m\ (A)\ \circ\ \lambda^m(B)$ … (6)

From (3), (4), and (6) gives

$\log \lambda^m\ (A)\ \circ\ \lambda^m(B) \uparrow < \log \lambda^m\ (AB) < \log \lambda(A^m B^m) < \log \lambda^m(A) \circ\ \lambda^m(B).$

The Proof is Completed.

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