ON INTUITIONISTIC FUZZY H-IDEALS IN BCI-ALGEBRAS

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Abstract
The aim of this paper is to introduce the notion of H-ideals in BCI- algebras and to investigate some of their properties..

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1.Introduction

The notion of BCK-algebras was introduced by Imai and Iseki in 1966.In the same year, Iseki[5] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. After the introduction of the concept of fuzzy sets by L.A. Zadeh [9] ,several researches were conducted on the generalization of the fuzzy sets. Y. B. Jun and J. Meng [7] introduced fuzzy p-ideals in BCI-algebras and studied several properties. H.M. Khalid and B.Ahmad [8] introduced fuzzy H-ideals in BCI-algebras.The idea of intuitionistic fuzzy set was first introduced by K.T.Atanassov [2,3], as a generalization of the notion of fuzzy set. In this paper using Atanassov's idea ,we establish the intuitionistic fuzzification of the concept of H-ideals in BCI-algebras and investigate some of their properties .

2.Preliminaries

In this section we include some elementary definitions that are necessary for this paper.

Definition 2.1 [4] An algebra (X , $*$,0) of type (2,0) is called a BCI-algebra if it satisfies the following axioms:

(1) $((x*y)*(x*z))*(z*y) = 0$,
(2) $(x*(x*y))*y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x ,y,z \in X$.

In a BCI-algebra X, we can define a partial ordering " $\leq$ " by putting $x\leq y$ if and only if $x*y = 0$

In a BCI- algebra X ,the following hold :

(5) $(x * z)* (y * z) \leq x*y$,
(6) $x *(x* (x * y)) = x*y$,
(7) $0*(x*y) = (0*x)*(0*y)$,
(8) $x*0 = x$,
(9) $(x*y)*z = (x*z)*y$,
(10) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$, for all $x ,y, z \in X$.

Example 2.2. The set X = { 0, 1, 2, 3 } with the following Cayley table is a BCI - algebra .

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Throughout this paper X always means a BCI-algebra without any specification.

Definition 2.3. A non- empty subset A of X is called an ideal of X if

(1) $0 \in A$,

(2) $x * y \in A$ and $y \in A$ imply $x \in A$.

Definition 2.4 . A non- empty subset A of X is called an H-ideal of X if

(3) $0 \in A$,

(4) $x * (y * z) \in A$ and $y \in A$ imply $x * z \in A$.

If we put z = 0,then it follows that A is an ideal.

Definition 2.5 [9]. Let X be a non-empty set. A fuzzy set μ in X is a function $\mu : X \to [\,0, 1\,]$.

Definition 2.6 [6] Let μ be a fuzzy set in X. For $t \in [0,1]$,the set $\mu_t = \{ x \in X | \mu(x) \geq t \}$ is called a level subset of μ.

Definition 2.7 [6] A fuzzy set μ in X is called a fuzzy ideal of X if for all $x, y \in X$ we have

(1) $\mu(0) \geq \mu(x)$,

(2) $\mu(x) \geq \min\{ \mu(x * y), \mu(y) \}$.

Definition 2.8 [6]. A fuzzy set μ in X is called a fuzzy H-ideal of X if for all $x, y, z \in X$,

(1) $\mu(0) \geq \mu(x)$,

(2) $\mu(x*z) \geq \min\{ \mu(x * (y * z)) , \mu(y) \}$.

Definition 2.9 [2]. An intuitionistic fuzzy set (IFS) A in a non empty set X is an object having the form A $= \{ < x, \mu_A(x), \nu_A(x) > / x \in X \}$, where the functions $\mu_A : X \to [\,0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership and the degree of non membership of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Notation: For the sake of simplicity, we shall use the symbol $A = < \mu_A, \nu_A >$ for the IFS $A = \{ < x, \mu_A(x), \nu_A(x) > / x \in X \}$.

Definition 2.10 [2]. Let A be an intuitionistic fuzzy set of a set X. For each pair $< t ,s > \in [0, 1]$, the set $A_{<t, s>} = \{ x \in X : \mu_A(x) \geq t$ and $\nu_A(x) \leq s \}$ is called the level subset of A .

Definition 2.11 [2]. Let A be an IFS in X and let $t \in [\,0, 1]$ .Then the sets $U(\mu_A ; t) = \{ x \in X : \mu_A(x) \geq t \}$and $L(\nu_A , t) = \{ x \in X : \nu_A(x) \leq t \}$ are called a μ -level t -cut and ν- level t-cut of A , respectively.

3. Intuitionistic fuzzy H-ideals

Definition 3.1 An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy H- ideal of X if for all x ,y∈X we have

(1) $\mu_A(0) \geq \mu_A(x)$,
(2) $\nu_A(0) \leq \nu_A(x)$,
(3) $\mu_A(x*z) \geq \min\{\mu_A(x*(y*z)), \mu_A(y)\}$,
(4) $\nu_A(x*z) \leq \max\{\nu_A(x*(y*z)), \nu_A(y)\}$.

Example 3.2. Let X = { 0,1,2,3 } with the following Cayley table be a BCI algebra.

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Let A= < μ_A, ν_A > be an IFS in X defined by
$\mu_A(0) = \mu_A(2) = 0.9$, $\mu_A(1) = \mu_A(3) = 0.09$ and $\nu_A(1) = \nu_A(3) = 0.9$ and $\nu_A(0) = \nu_A(2) = 0.09$. Then A is an intuitionistic fuzzy H-ideal of X.

Proposition 1. Every intuitionistic fuzzy H-ideal is an intuitionistic fuzzy ideal.

Theorem3.3. Let A=(μ_A, ν_A) in X be an intuitionistic fuzzy ideal of X. If $x \leq y$,then $\mu_A(x) \geq \mu_A(y)$,$\nu_A(x) \leq \nu_A(y)$, that is , μ_A is order reversing and ν_A is order preserving.

Proof. Let x ,y ∈X such that x ≤ y. Then

$x*y = 0$ and thus

$$\mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\} = \min\{\mu_A(0), \mu_A(y)\} = \mu_A(y).$$

and

$$\nu_A(x) \leq \max\{\nu_A(x*y), \nu_A(y)\} = \max\{\nu_A(0), \nu_A(y)\} = \nu_A(y).$$

Theorem 3.4. Let A= (μ_A, ν_A) in X be an intuitionistic fuzzy ideal of X. If $x*y \leq z$,then

$$\mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\}$$
$$\nu_A(x) \leq \max\{\nu_A(y), \nu_A(z)\}$$

Proof. Let x ,y , z ∈X such that $x*y \leq z$. Then

$(x*y)*z = 0$ and thus

$$\mu_A(x) \geq \min\{\mu_A(x*y), \mu_A(y)\} \geq \min\{\min\{\mu_A((x*y)*z), \mu_A(z)\}, \mu_A(y)\} = \min\{\min\{\mu_A(0), \mu_A(z)\}, \mu_A(y)\} = \min\{\mu_A(y), \mu_A(z)\}\}$$

and

$\nu_A(x) \le \max\{\nu_A(x* y), \nu_A(y)\}$
$\ge \max\{\max\{\nu_A((x* y)*z), \nu_A(z)\}, \nu_A(y)\}$
$= \max\{\max\{\nu_A(0), \nu_A(z)\}, \nu_A(y)\}$
$= \max\{\nu_A(y), \nu_A(z)\}\}$

Theorem 3.5. In an associative BCI-algebra X, every intuitionistic fuzzy ideal is an intuitionistic fuzzy H-ideal of X.

Proof. Let A be an intuitionistic fuzzy ideal of X. Then
(1) $\mu_A(0) \ge \mu_A(x)$,
(2) $\nu_A(0) \le \nu_A(x)$,

Since $\min\{\mu_A(x*(y*z)), \mu_A(y)\} = \min\{\mu_A(x*y)*z)), \mu_A(y)\}$
$= \min\{\mu_A((x*z)*y), \mu_A(y)\}$
$\le \mu_A(x*z)$
and
$\max\{\nu_A(x*(y*z)), \nu_A(y)\} = \max\{\nu_A(x*y)*z)), \nu_A(y)\}$
$= \max\{\nu_A((x*z)*y), \nu_A(y)\}$
$\ge \nu_A(x*z)$
Hence A is an intuitionistic fuzzy H - ideal of X.

Theorem 3.6. An intuitionistic fuzzy set A of a BCI-algebra X is an intuitionistic fuzzy H-ideal of X if and only if for any $t, s \in [0,1]$, the level subset $A_{<t,s>} = \{x \in X : \mu_A(x) \ge t \text{ and } \nu_A(x) \le s\}$ is either empty or an H-ideal of X.

Proof. Assume that A is an intuitionistic fuzzy H-ideal of X. Then
$\mu_A(0) \ge \mu_A(x)$ and $\nu_A(0) \le \nu_A(x)$. Therefore $\mu_A(0) \ge \mu_A(x) \ge t$
And $\nu_A(0) \le \nu_A(x) \le s$ for $t,s \in [0,1]$ or $\mu_A(0) \ge t$ and $\nu_A(0) \le s$ imply
$0 \in A_{<t,s>}$. Next, let $x*(y*z) \in A_{<t,s>}$ and $y \in A_{<t,s>}$. Then
$\mu_A(x*(y*z)) \ge t$, $\mu_A(y) \ge t$ and $\nu_A(x*(y*z)) \le s$, $\nu_A(y) \le s$. Since A is an intuitionistic fuzzy H-ideal of X,
$\mu_A(x*z) \ge \min\{\mu_A(x*(y*z)), \mu_A(y)\} \ge t$ and $\nu_A(x*z) \le \max\{\nu_A(x*(y*z)), \nu_A(y)\} \le s$
or $\mu_A(x*z) \ge t$ and $\nu_A(x*z) \le s$ imply $x*z \in A_{<t,s>}$. This proves that the level set $A_{<t,s>}$ is an H-ideal of X.
Conversely, we will show that A is an intuitionistic fuzzy ideal of X.
Suppose $\mu_A(x_0) > \mu_A(0)$
and $\nu_A(x_0) < \nu_A(0)$ for some $x_0 \in X$. Put $t_0 = ½[\mu_A(0) + \mu_A(x_0)]$ and
$s_0 = ½[\nu_A(0) + \nu_A(x_0)]$. Then $\mu_A(0) < t_0$ and $\nu_A(0) > s_0$ and $0 \le t_0 < \mu_A(x_0) \le 1$,
$1 \ge s_0 > \nu_A(x_0) \ge 0$. This implies $x_0 \in A_{<t_0,s_0>}$ or $A_{<t_0,s_0>} \ne \phi$. As $A_{<t_0,s_0>}$ is an H-ideal of X, we have $0 \in A_{<t_0,s_0>}$ and hence $\mu_A(0) \ge t_0$ and $\nu_A(x_0) \le s_0$. This contradiction proves that
$\mu_A(0) \ge \mu_A(x)$ and $\nu_A(x_0) \le \nu_A(x)$ for all $x \in X$. Next suppose for some x_0, y_0, z_0 in X
$\mu_A(x_0*z_0) < \min\{\mu_A(x_0*(y_0*z_0)), \mu_A(y_0)\}$ and $\nu_A(x_0*z_0) > \max\{\nu_A(x_0*(y_0*z_0)), \nu_A(y_0)\}$
Put $t_1 = 1/2[\mu_A(x_0*z_0) + \min\{\mu_A(x_0*(y_0*z_0)), \mu_A(y_0)\}]$ and
$s_1 = 1/2[\nu_A(x_0*z_0) + \max\{\nu_A(x_0*(y_0*z_0)), \nu_A(y_0)\}]$. Then
$\mu_A(x_0*z_0) < t_1$, $\nu_A(x_0*z_0) > s_1$ and $0 \le t_1 < \min\{\mu_A(x_0*(y_0*z_0)), \mu_A(y_0)\} \le 1$,
$1 \ge s_1 > \max\{\nu_A(x_0*(y_0*z_0)), \nu_A(y_0)\} \ge 0$ which gives $\mu_A(x_0*(y_0*z_0)) > t_1$, $\mu_A(y_0) > t_1$

and $\nu_A(x_0*(y_0*z_0)) < s_1$, $\nu_A(y_0) < s_1$ .Therefore $x_0*(y_0*z_0) \in A_{<t_1,s_1>}$, $y_0 \in A_{<t_1,s_1>}$ .Since $A_{<t_1,s_1>}$ is an H-ideal ,we have $x_0*z_0 \in A_{<t_1,s_1>}$ and hence $\mu_A(x_0*z_0) \geq t_1$ and $\nu_A(x_0*z_0) \leq s_1$, a contradiction .Hence the supposition is wrong .

This completes the proof.

Theorem 3.7. Let A be an intuitionistic fuzzy set of X. If A is an intuitionistic fuzzy H-ideal of X , then the set $X_A = \{ x \in X \mid \mu_A(x) = \mu_A(0)$ and $\nu_A(x) = \nu_A(0) \}$ is an H-ideal of X.

Proof. Assume that A is an intuitionistic fuzzy H-ideal of X. Clearly $0 \in X_A$.

Let $x*(y*z) \in X_A$, $y \in X_A$ .Since A is an intuitionistic fuzzy H-ideal in X,

$$\mu_A(x*z) \geq \min\{ \mu_A(x*(y*z)), \mu_A(y) \} = \min\{ \mu_A(0), \mu_A(0) \} = \mu_A(0)$$

Or $\mu_A(x*z) \geq \mu_A(0)$ .But $\mu_A(x*z) \leq \mu_A(0)$

and

$$\nu_A(x*z) \leq \max\{ \nu_A(x*(y*z)), \nu_A(y) \} = \max\{ \nu_A(0), \nu_A(0) \} = \nu_A(0)$$

Or $\nu_A(x*z) \leq \nu_A(0)$ .But $\nu_A(x*z) \geq \nu_A(0)$ .Hence $x*z \in X_A$ .This proves that X_A is an H-ideal of X.

Hence the proof.

Definition 3.8. A mapping f: $X \to Y$ is called a BCI- homomorphism if $f(x*y) = f(x)*f(y)$ for all x ,y in X .Note that in a BCI-homomorphism $f(0) = 0$

Theorem 3.9. Let f : **X** $\to$ Y be an onto BCI-homomorphism. If an intuitionistic fuzzy set B of Y is an intuitionistic fuzzy H - ideal , then the intuitionistic fuzzy subset $f^{-1}(B)$ of X is also an intuitionistic fuzzy H - ideal.

Proof. Let $y \in Y$. Since f is onto ,there exists $x \in X$ such that $y = f(x)$.Since B is an intuitionistic fuzzy H-ideal of Y ,it follows that $\mu_B(0) \geq \mu_B(y)$ and $\nu_B(0) \leq \nu_B(y)$ or

$\mu_B f(0) \geq \mu_B f(x)$ and $\nu_B f(0) \leq \nu_B f(x)$. Then by definition

$\mu_{f^{-1}(B)}(0) \geq \mu_{f^{-1}(B)}(x)$ and $\nu_{f^{-1}(B)}(0) \leq \nu_{f^{-1}(B)}(x)$ for all $x \in X$. Next , since B is an intuitionistic fuzzy H-ideal ,for any y_1,y_2,y_3 in Y , we have

$$\mu_B(y_1*y_3) \geq \min\{ \mu_B(y_1*(y_2*y_3)), \mu_B(y_2) \}$$
$$\mu_B(f(x_1)*f(x_3))\ (f(x_1*x_3)) \geq \min\{ \mu_B(f(x_1)*(f(x_2)*f(x_3))), \mu_B(x_2) \}$$
$$\mu_B(f(x_1*x_3)) \geq \min\{ \mu_B(f(x_1*(x_2*x_3)), \mu_B(x_2)) \}$$
$$\mu_{f^{-1}(B)}(x_1*x_3) \geq \min\{ \mu_{f^{-1}(B)}(x_1*(x_2*x_3)), \mu_{f^{-1}(B)}(x_2) \}$$

and

$$\nu_B(y_1*y_3) \leq \max\{ \nu_B(y_1*(y_2*y_3)), \nu_B(y_2) \}$$
$$\nu_B(f(x_1*x_3)) \leq \max\{ \nu_B(f(x_1)*(f(x_2)*f(x_3))), \nu_B(f(x_2)) \}$$
$$\nu_B(f(x_1)*f(x_3)) \leq \max\{ \nu_B(f(x_1*(x_2*x_3))), \nu_B(f(x_2)) \}$$
$$\nu_{f^{-1}(B)}(x_1*x_3) \leq \max\{ \nu_{f^{-1}(B)}(x_1*(x_2*x_3)), \nu_{f^{-1}(B)}(x_2) \}$$

which proves that $f^{-1}(B)$ is an intuitionistic fuzzy H-ideal of X.

This completes the proof.

Definition3.10. Let X and Y be nonempty sets and let f: X→Y be a mapping .Let $A = < \mu_A, \nu_A >$ be an IFS in X and $B= < \mu_B, \nu_B >$ be an IFS in Y .Then

(a) the preimage of B under f ,denoted by $f^{-1}(B)$,is the IFS in X defined by

$f^{-1}(B) = < f^{-1}(\mu_B), f^{-1}(\nu_B) >$,where $f^{-1}(\mu_B) = \mu_B \circ f$ and $f^{-1}(\nu_B) = \nu_B \circ f$

(b) the image of A under f ,denoted by f (A) ,is the IFS in Y defined by

$f(A) = < f(\mu_A), f(\nu_A) >$,where for each $y \in Y$

$$f(\mu_A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

and

$$f(\nu_A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

Theorem 3.11 . Let f: X→ Y be an onto BCI-homomorphism .If an intuitionistic fuzzy subset A of X is an intuitionistic fuzzy H-ideal ,then the intuitionistic fuzzy subset f(A) is also an intuitionistic fuzzy H-ideal of Y.

Proof. Since A is an intuitionistic fuzzy H-ideal of X **,** $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$

for all $x \in X$. Since $0 = f^{-1}(0)$,we have

$$\mu_{f(A)}(0) = \bigvee_{x \in f^{-1}(0)} \mu_A(x) = \mu_A(0) \geq \mu_A(x)$$

That is ,$\mu_{f(A)}(0) \geq \mu_A(x)$

$$\mu_{f(A)}(0) \geq \bigvee_{x \in f^{-1}(y)} \mu_A(x) = \mu_{f(A)}(y)$$

$\mu_{f(A)}(0) \geq \mu_{f(A)}(y)$ for all y in X

and

$$\nu_{f(A)}(0) = \bigwedge_{x \in f^{-1}(0)} \nu_A(x) = \nu_A(0) \leq \nu_A(x)$$

That is , $\nu_{f(A)}(0) \leq \nu_A(x)$

$$\nu_{f(A)}(0) \leq \bigwedge_{x \in f^{-1}(y)} \nu_A(x) = \nu_{f(A)}(y)$$

$\nu_{f(A)}(0) \leq \nu_{f(A)}(y)$ for all y in X .

Next ,let $y_1, y_2, y_3 \in Y$ and $x_1 \in f^{-1}(y_1)$, $x_2 \in f^{-1}(y_2)$, $x_3 \in f^{-1}(y_3)$ such that

$$\mu_A(x_1 * x_3) = \bigvee_{t \in f^{-1}(y_1 * y_3)} \mu_A(t),$$

$$\mu_A(x_2) = \bigvee_{t \in f^{-1}(y_2)} \{\mu_A(t), \mu_A(x_1 * (x_2 * x_3))\}$$

$$= \bigvee_{t \in f^{-1}(y_1 * (y_3 * y_3))} \mu_A(t).$$

$$\mu_{f(A)}(y_1 * y_3) = \bigvee_{t \in f^{-1}(y_1 * y_3)} \mu_A(t) = \mu_A(x_1 * x_2)$$

$$\geq \mu_A(x_1 * (x_2 * x_3)) \wedge \mu_A(x_2)$$

$$= \bigvee_{t\in f^{-1}(y_1*(y_3*y_3))} \mu_A(t) \wedge \bigvee_{t\in f^{-1}(y_2)} \mu_A(t).$$

$$= \mu_{f(A)}(y_1*(y_2*y_3)) \wedge \mu_{f(A)}(y_2)$$

and

$$\nu_A(x_1*x_3) = \bigwedge_{t\in f^{-1}(y_1*y_3)} \nu_A(t),$$

$$\nu_A(x_2) = \bigwedge_{t\in f^{-1}(y_2)} \{\nu_A(t), \nu_A(x_1*(x_2*x_3))\}$$

$$= \bigwedge_{t\in f^{-1}(y_1*(y_3*y_3))} \nu_A(t).$$

$$\nu_{f(A)}(y_1*y_3) = \bigwedge_{t\in f^{-1}(y_1*y_3)} \nu_A(t) = \nu_A(x_1*x_2)$$

$$\leq \nu_A(x_1*(x_2*x_3)) \wedge \nu_A(x_2)$$

$$= \bigwedge_{t\in f^{-1}(y_1*(y_3*y_3))} \nu_A(t) \vee \bigwedge_{t\in f^{-1}(y_2)} \nu_A(t).$$

$$= \nu_{f(A)}(y_1*(y_2*y_3)) \vee \nu_{f(A)}(y_2)$$

This proves that f(A) is also an intuitionistic fuzzy H-ideal of Y.
This completes the proof.

Definition 3.12[4]. A fuzzy relation A on any set X is a fuzzy subset A with a membership function $\mu_A : X \times X \to [0,1]$

Definition 3.13[4] . If A is a fuzzy relation with membership function μ_A on a set X and B is a fuzzy subset of X with membership function μ_B ,then A is a fuzzy relation on B ,if for all $x, y \in X$, $\mu_A(x,y) \leq \mu_B(x) \wedge \mu_B(y)$.

Lemma 3.14[4]. Let A and B be fuzzy sets of a set X with membership functions μ_A and μ_B , respectively. Then the Cartesian product A×B of A and B is a fuzzy relation on a set X whose membership function $\mu_{A\times B}$ is defined as $\mu_{A\times B}(x,y) = \mu_A(x) \wedge \mu_B(y)$ for all $x, y \in X$.

Definition 3.15[4]. Let A and B be intuitionistic fuzzy sets of a set X. Then the Cartesian product A×B of A and B is a fuzzy relation on a set X is defined as
$\mu_{A\times B}(x,y) = \mu_A(x) \wedge \mu_B(y)$ and $\nu_{A\times B}(x,y) = \nu_A(x) \vee \nu_B(y)$ for all $x, y \in X$.

Theorem 3.16. Let A and B be intuitionistic fuzzy H-ideals of X. Then the Cartesian product A×B of A and B is an intuitionistic fuzzy H-ideal of X×X.

Proof. Let **$(x, y) \in X$.**Then by definition

$$\mu_{A\times B}(0,0) = \mu_A(0) \wedge \mu_B(0)$$

$$\geq \mu_A(x) \wedge \mu_B(y) = \mu_{A\times B}(x,y)$$

and

$$\nu_{A\times B}(0,0) = \nu_A(0) \vee \nu_B(0)$$

$$\leq \nu_A(x) \vee \nu_B(y) = \nu_{A\times B}(x,y)$$

Next , consider $\mu_{A\times B}((x_1 ,x_2)* ((y_1 ,y_2)*(z_1, z_2))) \wedge \mu_{A\times B}((y_1 ,y_2)$
$= \mu_{A\times B}((x_1 ,x_2)* (y_1*z_1 , y_2*z_2)) \wedge \mu_{A\times B}(y_1 ,y_2)$
$= \mu_{A\times B}(x_1* (y_1*z_1) , x_2*(y_2*z_2)) \wedge \mu_{A\times B}((y_1 ,y_2)$
$= [\mu_A (x_1* (y_1*z_1)) \wedge \mu_B(x_2*(y_2*z_2))] \wedge [\mu_A (y_1) \wedge \mu_B(y_2)]$
$= [\mu_A (x_1* (y_1*z_1)) \wedge \mu_A (y_1)] \wedge[\mu_B(x_2*(y_2*z_2)) \wedge \mu_B(y_2)]$
$\leq \mu_A (x_1* z_1) \wedge \mu_B(x_2*y_2)$
$= \mu_{A\times B}(x_1* z_1 , x_2*z_2)$
$= \mu_{A\times B}((x_1 ,x_2) ,(z_1 , z_2))$

and

$\nu_{A\times B}((x_1 ,x_2)*((y_1 ,y_2)*(z_1, z_2))) \vee \nu_{A\times B}((y_1 ,y_2)$
$= \nu_{A\times B}((x_1 ,x_2)* (y_1*z_1 , y_2*z_2)) \vee \nu_{A\times B}(y_1 ,y_2)$
$= \nu_{A\times B}(x_1* (y_1*z_1) , x_2*(y_2*z_2)) \vee \nu_{A\times B}((y_1 ,y_2)$
$= [\nu_A (x_1* (y_1*z_1)) \vee \nu_B(x_2*(y_2*z_2))] \vee [\nu_A (y_1) \vee \nu_B(y_2)]$
$= [\nu_A (x_1* (y_1*z_1)) \vee \nu_A (y_1)] \vee [\nu_B(x_2*(y_2*z_2)) \vee \nu_B(y_2)]$
$\geq\nu_A (x_1* z_1) \vee \nu_B(x_2*y_2)$
$= \nu_{A\times B}(x_1* z_1 , x_2*z_2)$
$= \nu_{A\times B}((x_1 ,x_2) ,(z_1 , z_2))$

This completes the proof.

Theorem 3.17. Let A and B be intuitionistic fuzzy sets of a BCI-algebra X .If **A**×B is an intuitionistic fuzzy H-ideal of X ×X, then
(i) $\mu_A(0) \geq\mu_A(x)$ and $\nu_A(0) \leq\nu_A(x)$ or $\mu_B(0) \geq\mu_B(x)$ and $\nu_B(0) \leq\nu_B(x)$
(ii) $\mu_B(0) \geq\mu_A(x)$ and $\nu_B(0) \leq\nu_A(x)$ or $\mu_B(0) \geq\mu_B(x)$ and $\nu_B(0) \leq\nu_B(x)$
(iii) $\mu_A(0) \geq\mu_A(x)$ and $\nu_A(0) \leq\nu_A(x)$ or $\mu_A(0) \geq\mu_B(x)$ and $\nu_A(0) \leq\nu_B(x)$
(iv) A or B is an intuitionistic fuzzy H-ideal of X.

Proof. (i) Suppose $\mu_A(0) < \mu_A(x)$, $\nu_A(0) >\nu_A(x)$ and $\mu_B(0) <\mu_B(y)$, $\nu_B(0) >\nu_B(y)$ for some x ,y in X.
Then $\mu_{A\times B}(x ,y) =\mu_A(x) \wedge\mu_B(y) > \mu_A(0) \wedge\mu_B(0) =\mu_{A\times B}(0 ,0)$ and
$\nu_{A\times B}(x ,y) =\nu_A(x) \vee\nu_B(y) < \nu_A(0) \vee\nu_B(0) =\nu_{A\times B}(0 ,0)$ or
$\mu_{A\times B}(x ,y) > \mu_{A\times B}(0 ,0)$and $\nu_{A\times B}(x ,y) < \nu_{A\times B}(0 ,0)$ for all x ,y $\in$ X×X ,a contradiction.
This proves (i)
(ii) Suppose $\mu_B(0) <\mu_A(x)$, $\nu_B(0) >\nu_A(x)$ and $\mu_B(0) <\mu_B(y)$, $\nu_B(0) > \nu_B(y)$ for some x ,y in X. Then $\mu_{A\times B}(0 ,0) =\mu_A(0) \wedge\mu_B(0)= \mu_B(0)$ and $\nu_{A\times B}(0 ,0) =\nu_A(0) \vee \nu_B(0)= \nu_B(0)$
It follows that $\mu_{A\times B}(x ,y) =\mu_A(x) \wedge\mu_B(y) > \mu_B(0) \wedge\mu_B(0) = \mu_B(0)= \mu_{A\times B}(0 ,0)$ or
$\mu_{A\times B}(x ,y)> \mu_{A\times B}(0 ,0)$
and
$\nu_{A\times B}(x ,y) =\nu_A(x) \vee \nu_B(y) < \nu_B(0) \vee\nu_B(0) = \nu_B(0)= \nu_{A\times B}(0 ,0)$ or
$\nu_{A\times B}(x ,y)< \nu_{A\times B}(0 ,0)$ which is a contradiction .This proves (ii)
(iii) is similar to (ii)
(iv) By (i) suppose $\mu_B(0) \geq\mu_B(x)$ and $\nu_B(0) \leq\nu_B(x)$ for all x $\in$X.
From (iii), take $\mu_B(x) \leq \mu_A(0)$ and $\nu_B(x) \geq\nu_A(0)$ for all x $\in$X. Then
$\mu_{A\times B}(0 ,x) =\mu_A(0) \wedge\mu_B(x)= \mu_B(x)$ and $\nu_{A\times B}(0 ,x) =\nu_A(0) \vee \nu_B(x)= \nu_B(x)$ (I)
Since A×B is an intutionistic fuzzy H-ideal ,we have
$\mu_{A\times B}((x_1 , x_2)* (z_1 , z_2)) \geq\mu_{A\times B}((x_1 , x_2)*((y_1 , y_2) *(z_1 , z_2))) \wedge\mu_{A\times B}(y_1 , y_2)$
or
$\mu_{A\times B}(x_1* z_1 ,x_2* z_2)) \geq\mu_{A\times B}(x_1 *(y_1*z_1), x_2*(y_2 * z_2)) \wedge\mu_{A\times B}(y_1 , y_2)$
and

$\nu_{A\times B}((x_1, x_2)*(z_1, z_2)) \leq \nu_{A\times B}((x_1, x_2)*((y_1, y_2)*(z_1, z_2))) \vee \nu_{A\times B}(y_1, y_2)$

or

$\nu_{A\times B}(x_1* z_1, x_2* z_2)) \leq \nu_{A\times B}(x_1*(y_1*z_1), x_2*(y_2*z_2)) \vee \nu_{A\times B}(y_1, y_2)$

If $x_1= y_1 = z_1= 0$,then

$\mu_{A\times B}(0, x_2* z_2)) \geq \mu_{A\times B}(0, x_2*(y_2*z_2)) \wedge \mu_{A\times B}(0, y_2)$ and

$\nu_{A\times B}(0, x_2* z_2)) \leq \nu_{A\times B}(0, x_2*(y_2*z_2)) \vee \nu_{A\times B}(0, y_2)$

By (I) we have

$\mu_{A\times B}(x_2* z_2) \geq \mu_{A\times B}(x_2*(y_2*z_2)) \wedge \mu_{A\times B}(y_2)$ and

$\nu_{A\times B}(x_2* z_2)) \leq \nu_{A\times B}(x_2*(y_2*z_2)) \vee \nu_{A\times B}(y_2)$

This proves that B is an intuitionistiic fuzzy H-ideal of X. Similar is the case when $\mu_A(x) \leq \mu_A(0)$, $\nu_A(x) \geq \nu_A(0)$ and $\mu_A(x) \leq \mu_B(0)$, $\nu_A(x) \geq \nu_B(0)$ for all $x\in X$. This gives that A is an intuitionistic fuzzy H - ideal of X.

This completes the proof.

Definition 3.18[4]. Let B be a fuzzy subset of X. Then the strongest fuzzy relation on X, that is, a fuzzy relation A on B whose membership function μ_{A_B} is given by

$\mu_{A_B}(x, y) = \mu_B(x) \wedge \mu_B(y)$ for all $x, y \in X$

Definition 3.19. Let A and B be two intuitionistic fuzzy subsets of X. Then the strongest intuitionistic fuzzy relation on X, that is,an intuitionistic fuzzy relation A on B is A_B defined by $\mu_{A_B}(x, y) = \mu_B(x) \wedge \mu_B(y)$and

$\nu_{A_B}(x, y) = \nu_B(x) \vee \nu_B(y)$ for all $x, y \in X$

Theorem 3.20. Let B be an intuitionistic fuzzy subset of a BCI-algebra X and A_B be the strongest intuitionistic fuzzy relation on X. Then B is an intuitionistic fuzzy H-ideal of X if and only if A_B is an intuitionistic fuzzy H-ideal of X×X.

Proof. Assume that B is an intuitionistic fuzzy H-ideal of X. Then

$\mu_{A_B}(0, 0) = \mu_B(0) \wedge \mu_B(0) \geq \mu_B(x) \wedge \mu_B(y) = \mu_{A_B}(x, y)$ and

$\nu_{A_B}(0, 0) = \nu_B(0) \vee \nu_B(0) \leq \nu_B(x) \vee \nu_B(y) = \nu_{A_B}(x, y)$ for all (x, y) in X×X. Next,

$$\begin{aligned}\mu_{A_B}((x_1, x_2)*(z_1, z_2)) = \mu_{A_B}(x_1* z_1, x_2*z_2) &= \mu_B(x_1* z_1) \wedge \mu_B(x_2*z_2)\\ &\geq [\mu_B(x_1*(y_1* z_1)) \wedge \mu_B(y_1)] \wedge [\mu_B(x_2*(y_2* z_2)) \wedge \mu_B(y_2)]\\ &= [\mu_B(x_1*(y_1* z_1)) \wedge \mu_B(x_2*(y_2* z_2))] \wedge [\mu_B(y_1) \wedge \mu_B(y_2)]\\ &= \mu_{A_B}(x_1*(y_1*z_1), x_2*(y_1, z_2)) \wedge \mu_{A_B}(y_1, y_2)\end{aligned}$$

and

$$\begin{aligned}\nu_{A_B}((x_1, x_2)*(z_1, z_2)) = \nu_{A_B}(x_1* z_1, x_2*z_2) &= \nu_B(x_1* z_1) \vee \nu_B(x_2*z_2)\\ &\leq [\nu_B(x_1*(y_1* z_1)) \vee \nu_B(y_1)] \vee [\nu_B(x_2*(y_2* z_2)) \vee \nu_B(y_2)]\\ &= [\nu_B(x_1*(y_1* z_1)) \vee \nu_B(x_2*(y_2* z_2))] \vee [\nu_B(y_1) \vee \nu_B(y_2)]\\ &= \nu_{A_B}(x_1*(y_1*z_1), x_2*(y_1, z_2)) \vee \nu_{A_B}(y_1, y_2)\end{aligned}$$

for all (x_1,x_2), (y_1,y_2), (z_1,z_2) in X×X. This proves that A_B is an intuitionistic fuzzy H-ideal of X×X. Conversely, let A_B be an intuitionistic fuzzy H-ideal of X×X. Then for all $(x .x) \in X\times X$

$\mu_B(0) \wedge \mu_B(0) = \mu_{A_B}(0,0) > \mu_{A_B}(x,x) = \mu_B(x) \wedge \mu_B(x)$ or $\mu_B(0) > \mu_B(x)$ and

$\nu_B(0) \vee \nu_B(0) = \nu_{A_B}(0,0) < \nu_{A_B}(x,x) = \nu_B(x) \vee \nu_B(x)$ or $\nu_B(0) < \nu_B(x)$

for all $x \in X$. Next ,let (x_1,x_2), (y_1,y_2), $(z_1,z_2) \in X \times X$. Then

$$\begin{aligned}
\mu_B(x_1 * z_1) \wedge \mu_B(x_2 * z_2) &= \mu_{A_B}(x_1 * z_1, x_2 * z_2) = \mu_{A_B}((x_1,x_2), (z_1,z_2)) \\
&\geq \mu_{A_B}((x_1,x_2) * ((y_1,y_2) * (z_1,z_2))) \wedge \mu_{A_B}(y_1,y_2) \\
&= \mu_{A_B}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) \wedge \mu_{A_B}(y_1,y_2) \\
&= [\mu_B(x_1 * (y_1 * z_1)) \wedge \mu_B(x_2 * (y_2 * z_2))] \wedge [\mu_B(y_1) \wedge \mu_B(y_2)] \\
&= [\mu_B(x_1 * (y_1 * z_1)) \wedge \mu_B(y_1)] \wedge [\mu_B(x_2 * (y_2 * z_2)) \wedge \mu_B(y_2)]
\end{aligned}$$

and

$$\begin{aligned}
\nu_B(x_1 * z_1) \vee \nu_B(x_2 * z_2) &= \nu_{A_B}(x_1 * z_1, x_2 * z_2) = \nu_{A_B}((x_1,x_2), (z_1,z_2)) \\
&\leq \nu_{A_B}((x_1,x_2) * ((y_1,y_2) * (z_1,z_2))) \vee \nu_{A_B}(y_1,y_2) \\
&= \nu_{A_B}(x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) \vee \nu_{A_B}(y_1,y_2) \\
&= [\nu_B(x_1 * (y_1 * z_1)) \vee \nu_B(x_2 * (y_2 * z_2))] \vee [\nu_B(y_1) \vee \nu_B(y_2)] \\
&= [\nu_B(x_1 * (y_1 * z_1)) \vee \nu_B(y_1)] \vee [\nu_B(x_2 * (y_2 * z_2)) \vee \nu_B(y_2)]
\end{aligned}$$

Taking $x_2 = y_2 = z_2 = 0$,we get

$\mu_B(x_1 * z_1)) \wedge \mu_B(0) \geq \mu_B(x_1 * (y_1 * z_1)) \wedge \mu_B(y_1) \wedge \mu_B(0)$ or

$\mu_B(x_1 * z_1)) \geq \mu_B(x_1 * (y_1 * z_1)) \wedge \mu_B(y_1)$

and

$\nu_B(x_1 * z_1)) \vee \nu_B(0) \leq \nu_B(x_1 * (y_1 * z_1)) \vee \nu_B(y_1) \vee \nu_B(0)$ or

$\nu_B(x_1 * z_1)) \leq \nu_B(x_1 * (y_1 * z_1)) \vee \nu_B(y_1)$

which proves that B is an intuitionistic fuzzy H-ideal of X.
This completes the proof.

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