COMMON FIXED POINT THEOREM IN INTUTIONISTIC FUZZY METRIC SPACE USING ABSORBING MAPS

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Abstract

The aim of this paper is to introduce the new notion of absorbing maps in intuitionistic fuzzy metric space which is neither a subclass of compatible maps nor a subclass of non-compatible maps, it is not necessary that absorbing maps commute at their coincidence points however if the mapping pair satisfy the contractive type condition then point wise absorbing maps not only commute at their coincidence points but it becomes a necessary condition for obtaining a common fixed point of mapping pair.

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Key words : Fixed point, common fixed point, fuzzy metric space, intutionistic metric space, absorbing mapping.

1. Introduction

In 2004, Park [8] introduced a notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [2] in fact the concepts of triangular norms (t-norm) and triangular conorms (t-conorm) are originally introduced by Schweizer and Sklar [10] in study of statistical metric spaces.

Ranadive et. al. [9] introduced the concept of absorbing mapping in metric space and prove common fixed point theorem in this space. Moreover they observe that the new notion of absorbing map is neither a subclass of compatible maps nor a subclass of non compatible maps. In [4] Mishra et. al. introduced absorbing maps in fuzzy metric space.

Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. Pant [5,6,7] noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in metric spaces. They also showed that in the setting of common fixed point theorems for compatible mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings. They also showed that in the setting of common fixed point theorems for compatible *mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings.*

Balasubramaniam et.al. [1] proved a fixed point theorem, which generalizes a result of Pant [6] for self mappings in fuzzy metric space. Pant and Jha [7] proved a fixed point theorem that gives an analogue of the results by Balasubramaniam et.al.[1] by obtaining a connection between the continuity and reciprocal continuity for four mappings in fuzzy metric space.

Kumar and Chugh [3] established some common fixed point theorems in metric spaces by using the ideas of pointwise R-weak commutativity and reciprocal continuity of mappings.

2. Preliminaries

In this section we give some definitions which are used to prove of our main results.

Definition 2.1. *Let X be any non empty set. A fuzzy set A in X is a function with domain X and values in* $[0,1]$.

Definition 2.2 *Let a set E be fixed. An intuitionistic fuzzy set (IFS) A of E is an object having the form* $A = \{< x, \mu_A(x), \nu_A(x) >: x \in E\}$ *where the function* $\mu_A : E \to [0,1]$ *and* $\nu_A : E \to [0,1]$, *define respectively, the degree of membership and degree of non-membership of the element* $x \in E$ *to the set* A, *which is a subset of E, and for every* $x \in E$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 2..3 *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm, if * is satisfying the following conditions :*

*2.3 (i) * is commutative and associative.*

*2.3 (ii) * is continuous.*

2.3 (iii) $a * 1 = a$ *for all* $a \in [0,1]$.

2.3 (iv) $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$,

For $a, b, c, d \in [0,1]$.

Definition 2..4 *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is continuous t-conorm if* $\diamond$ *it satisfies the following conditions:*

2.4 (i) $\diamond$ *is commutative and associative.*

2.4 (ii) $\diamond$ *is continuous.*

2.4 (iii) $a \diamond 0 = a$ *for all* $a \in [0, 1]$.

2.4 (iv) $a \diamond b \le c \diamond d$ *whenever* $a \le c$ *and* $b \le d$,

For $a, b, c, d \in [0,1]$.

Note 2.5 *The concepts of triangular norms (t-norms) and triangular co norms (t-co norms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [102] in his study of statistical metric spaces.*

Definition 2.6 *A 5-tuple* $(X, M, N, *, \diamond)$ *is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if X is an arbitrary set,* $*$ *is a continuous t-norm is a continuous t-conorm and* M, N *are fuzzy sets on* $X^2 \times (0, \infty)$ *satisfying the following conditions: for all* $x, y, z, s, t > 0$,

2.6 (IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$

2.6 (IFM-2) $M(x, y, 0) = 0$

2.6 (IFM-3) $M(x, y, t) = 1$ *if and only if* $x = y$.

2.6 (IFM-4) $M(x, y, t) = M(y, x, t)$

2.6 (IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

2.6 (IFM-6) $M(x, y, \bullet): [0, \infty) \rightarrow [0,1]$ *is left continuous.*

2.6 (IFM-7) $\lim_{t \rightarrow \infty} M(x, y, t) = 1$

2.6 (IFM-8) $N(x, y, 0) = 1$

2.6 (IFM-9) $N(x, y, t) = 0$ *if and only if* $x = y$.

2.6 (IFM-10) $N(x, y, t) = N(y, x, t)$

2.6 (IFM-11) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$

2.6 (IFM-12) $N(x, y, \bullet): [0, \infty) \rightarrow [0,1]$ *is right continuous.*

2.6 (IFM-13) $\lim_{t \rightarrow \infty} N(x, y, t) = 0$

Then (M, N) *is called an intuitionistic fuzzy metric on X. The functions* $M(x, y, t)$ *and* $N(x, y, t)$ *denote the degree of nearness and degree of non-nearness between* x *and* y *with respect to t, respectively.*

Remark 2.7 *Every fuzzy metric space* $(X, M, *)$ *is an intuitionistic fuzzy metric space if X of the form* $(X, M, 1 - M, *, \diamond)$ *such that t- norm* $*$ *and t-conorm* $\diamond$ *are associated, that is,*

$$x \diamond y = 1 - ((1 - x) * (1 - y)) \text{ for any } x, y \in X$$

but the converse is not true.

Example 2.8 *(Induced intuitionistic fuzzy metric). Let (X,d) be a metric space. Define* $a * b = a\,b$ *and* $a \diamond b = \min\{1, a + b\}$ *for all* $a, b \in [0, 1]$ *and let* M_d *and* N_d *be fuzzy sets on* $X^2 \times (0, \infty)$ *defined as follows:*

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x,y)} \quad \text{and} \quad N_d(x, y, t) = \frac{d(x,y)}{kt^n + md(x,y)} .$$

for all h, k, m *and* $n \in R^+$. *Then* $(X, M_d, N_d, *, \diamond)$ *is an intuitionistic fuzzy metric space.*

Remark 2.9 *Note that the above example holds even with the t-norm*

$a * b = min\{a, b\}$ *and* t – *conorm* $a \diamond b = max\{a, b\}$ *and hence* (M, N) *is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm.*

In the above example by taking $h = k = m = n = 1$, *we get*

$$M_d(x, y, t) = \frac{t}{t + d(x,y)} \text{ and } N_d(x, y, t) = \frac{d(x,y)}{t + d(x,y)}$$

We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Example 2.10 *Let* $X = N$. *Define* $a * b = max\{0, a + b - 1\}$ *and* $a \diamond b = (a + b - ab)$ *for all* $a, b \in [0,1]$ *and let* M *and* N *be fuzzy sets on* $X^2 \times (0, \infty)$ *defined as Follows:*

$$M(x, y, t) = \begin{cases} \frac{x}{y}, & if\ x \le y \\ \frac{y}{x}, & if\ y \le x \end{cases} \quad and \quad N(x, y, t) = \begin{cases} \frac{y-x}{y}, & if\ x \le y \\ \frac{x-y}{x}, & if\ y \le x \end{cases}$$

for all $x, y, z \in X$ *and* $t > 0$. *Then* $(X, M, N, *, \diamond)$ *is an intuitionistic fuzzy metric space.*

Remark 2.11 *Note that, in the above example, t-norm* $*$ *and t- conorm* $\diamond$ *are not associated, and there exists no metric d on X satisfying*

$$M(x, y, t) = \frac{t}{t + d(x,y)} \text{ and } N(x, y, t) = \frac{d(x,y)}{t + d(x,y)}$$

Where $M(x, y, t)$ *and* $N(x, y, t)$ *are as defined in above example. Also note that the above functions* (M, N) *is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as* $a * b = min\{a, b\}$ *and* t – *conorm* $a \diamond b = max\{a, b\}$.

Definition 2.12 *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space.*

(a) *A sequence* $\{x_n\}$ *in X is called cauchy sequence if for each* $t > 0$ *and* $p > 0$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ *and* $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$.
(b) *A sequence* $\{x_n\}$ *in X is convergent to* $x \in X$ *if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *and* $\lim_{n\to\infty} N(x_n, x, t) = 0$, *for each* $t > 0$.
(c) *An intuitionistic fuzzy metric space is said to be complete if every Cauchy sequence is convergent.*

Definition 2.13 *Let A and B be mappings from an intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *into itself. Then the mappings are said to be reciprocally continuous if* $\lim_{n\to\infty} ABx_n = Az$ *and* $\lim_{n\to\infty} BAx_n = Bz$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} ABx_n = \lim_{n\to\infty} BAx_n = z$, *for some* $z \in X$.

Remark 2.14 *If A and B are both continuous then they are obviously reciprocally continuous. But the converse need not be true.*

Example 2.15 *Let* $X = [4,30]$ *and* d *be the usual metric space* X. *Define mappings* $A, B: X \to X$ *by*

$$Ax = \begin{cases} x & if\ x = 4 \\ 13 & if\ x > 4 \end{cases} \quad and \quad Bx = \begin{cases} x & if\ x = 4 \\ 26 & if\ x > 4 \end{cases}$$

It may be noted that A and B are reciprocally continuous mappings but neither A nor B is continuous mappings.

We shall use the following lemmas to prove our next result without any further citation:

Lemma 2.16 *In an intuitionistic fuzzy metric space X, $M\,(x, y, .)$ is non-decreasing and $N\,(x, y, .)$ is non increasing for all $x, y \in X$.*

Lemma 2.17 *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. If there exists a constant $k \in (0\,, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) = M(y_{n+1}, y_n, t)$$

And

$$N(y_{n+2}, y_{n+1}, kt) = N(y_{n+1}, y_n, t)$$

For every $t > 0$ and n =1, 2, … . Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.18 *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. If there exists a constant $k \in (0,1)$ such that*

$$M(x, y, kt) = M(x, y, t) \ \ and\ \ N(x, y, kt) = N(x, y, t)$$

for $x, y \in X$. Then $x = y$.

Definition 2.19 *Let $\mathcal{A}$ and $\mathcal{B}$ are two self maps on a intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ then $\mathcal{A}$ is called $\mathcal{B} -$ absorbing if there exists a positive integer $R > 0$ such that*

$$M(\mathcal{B}\,x, \mathcal{B}\mathcal{A}\,x, t) \geq M(\mathcal{B}\,x, \mathcal{A}\,x, t/R)$$

$$N(\mathcal{B}\,x, \mathcal{B}\mathcal{A}\,x, t) \leq N(\mathcal{B}\,x, \mathcal{A}\,x, t/R)$$

for all $x \in X$. Similarly $\mathcal{B}$ is called $\mathcal{A}$ - absorbing if there exists a positive integer $R > 0$ such that

$$M(\mathcal{A}\,x, \mathcal{A}\,\mathcal{B}\,x, t) \geq M(\mathcal{A}\,x, \mathcal{B}\,x, t/R)$$

$$N(\mathcal{A}\,x, \mathcal{A}\,\mathcal{B}\,x, t) \leq N(\mathcal{A}\,x, \mathcal{B}\,x, t/R)$$

for all $x \in X$.

Definition 2.20 *The map $\mathcal{A}$ is called point wise $\mathcal{B}$ - absorbing if for given $x \in X$, there exists a positive integer $R > 0$ such that*

$$M(\mathcal{B}\,x, \mathcal{B}\mathcal{A}\,x, t) \geq M(\mathcal{B}\,x, \mathcal{A}\,x, t/R)$$

$$N(\mathcal{B}\,x, \mathcal{B}\mathcal{A}\,x, t) \leq N(\mathcal{B}\,x, \mathcal{A}\,x, t/R)$$

for all $x \in X$ *. Similarly we can defined point wise* $\mathcal{A}$ *- absorbing maps.*

Example 2.21 *Let* (X, d) *be usual metric space where* $X = [2, 20]$ *and* (M, N) *be the usual intuitionistic fuzzy metric on* $(X, M, N, *, \diamond)$ *with*

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0 \\ 0, & t = 0 \end{cases} \quad and \quad N(x, y, t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & t > 0 \\ 1, & t = 0 \end{cases}$$

For $x, y \in X$ *,* $t > 0$*. We define*

$$A(x) = \begin{cases} 6 & if\ 2 \le x \le 5;\ and\ x = 6 \\ 10 & if\ x > 6 \\ \frac{x-1}{2} & if\ x \in (5,6) \end{cases}$$

and

$$B(x) = \begin{cases} 2 & if\ 2 \le x \le 5 \\ \frac{x+1}{3} & if\ x > 5 \end{cases}$$

If we choose $x_n = 5 + \frac{1}{2n}$ *for n = 1,2,3...... then both pairs* (A, B) *and* (B, A) *are not compatible but A is B-absorbing and B is A-absorbing.*

3. Main Results

Theorem 3.1 *Let* P *be point wise* S *- absorbing and* Q *be point wise* T *– absorbing self maps on a complete intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *with continuous t-norm defined by* $a * b = min\{a, b\}$ *and* $a \diamond b = max\{a, b\}$ *where* $a, b \in (0, 1)$*, satisfying the conditions:*

3.1(I) $P(X) \subset T(X), Q(X) \subset S(X)$

3.1(II) There exists $k \in (0,1)$ *such that for every* $x, y \in X$ *and* $t > 0$*,*

$$M(Px, Qy, kt) \ge min\begin{Bmatrix} M(Sx, Ty, t), M(Px, Sx, t), M(Qy, Ty, t), \\ M(Px, Ty, t) \end{Bmatrix}$$

$$N(Px, Qy, kt) \le min\begin{Bmatrix} N(Sx, Ty, t), N(Px, Sx, t), N(Qy, Ty, t), \\ N(Px, Ty, t) \end{Bmatrix}$$

3.1(III) for all $, y \in X$ *,* $\lim_{t\to\infty} M(x, y, t) = 1$ *and* $\lim_{t\to\infty} N(x, y, t) = 0$

If the pair of maps (P, S) *is reciprocal continuous compatible maps then* P, Q, S *and* T *have a unique common fixed point in* X*.*

Proof: *let* x_0 *be any arbitrary point in* X*, construct a sequence* $y_n \in X$ *such that*

$$y_{2n-1} = Tx_{2n-1} = Px_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Qx_{2n+1},\ n = 1, 2, 3, \ldots \quad (1)$$

This can be done by the virtue of 2.2.1(I). By using contractive condition we obtain,

$$M\,(y_{2n+1}, y_{2n+2}, kt) = M(Px_{2n}, Qx_{2n+1}, kt)$$
$$\geq min\begin{Bmatrix} M(Sx_{2n}, Tx_{2n+1}, t), M\,(Px_{2n}, Sx_{2n}, t), \\ M\,(Qx_{2n+1}, Tx_{2n+1}, t), M(P\,x_{2n}, T\,x_{2n+1}, t) \end{Bmatrix}$$

$$\geq min\begin{Bmatrix} M(y_{2n}, y_{2n+1}, t), M\,(y_{2n+1}, y_{2n}, t), \\ M\,(y_{2n}, y_{2n+1}, t), 1 \end{Bmatrix}$$

$$N\,(y_{2n+1}, y_{2n+2}, kt) = N(Px_{2n}, Qx_{2n+1}, kt)$$

$$\leq min\begin{Bmatrix} N(Sx_{2n}, Tx_{2n+1}, t), N\,(Px_{2n}, Sx_{2n}, t), \\ N\,(Qx_{2n+1}, Tx_{2n+1}, t), N(P\,x_{2n}, T\,x_{2n+1}, t) \end{Bmatrix}$$

$$\leq min\begin{Bmatrix} N(y_{2n}, y_{2n+1}, t), N\,(y_{2n+1}, y_{2n}, t), \\ N\,(y_{2n}, y_{2n+1}, t), 0 \end{Bmatrix}$$

Which implies,

$$M\,(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)$$

$$N\,(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t)$$

in general $$M\,(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$$

$$N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t) \qquad (2)$$

To prove $\{y_n\}$ *is a Cauchy sequence, we have to show*

$$M\,(y_n, y_{n+1}, t) \to 1 \quad and\, N\,(y_n, y_{n+1}, t) \to 0$$

(for $t > 0$ *as* $n \to \infty$ *uniformly on* $p \in N$ *),for this from (1) we have,*

$$M\,(y_n, y_{n+1}, t) \geq M\left(y_{n-1}, y_n, \tfrac{t}{k}\right) \geq \;\dots\; \geq M\left(y_0, y_1, \tfrac{t}{k^n}\right) \to 1$$

$$N\,(y_n, y_{n+1}, t) \geq N\left(y_{n-1}, y_n, \tfrac{t}{k}\right) \geq \;\dots\; \geq N\left(y_0, y_1, \tfrac{t}{k^n}\right) \to 0$$

As $n \to \infty$ *for* $p \in N$ *, by (1) we have*

$$M\left(y_n, y_{n+p}, t\right) \geq M(y_n, y_{n+1}, (1-k)t) * M\left(y_{n+1}, y_{n+p}, kt\right)$$

$$\geq M\left(y_0, y_1, \tfrac{(1-k)t}{k^n}\right) * M\,(y_{n+1}, y_{n+2}, t) * M\left(y_{n+2}, y_{n+p}, (k-1)t\right)$$

$$\geq M\left(y_0, y_1, \tfrac{(1-k)t}{k^n}\right) * M\left(y_0, y_1, \tfrac{t}{k^n}\right) * M\,(y_{n+2}, y_{n+3}, t)$$

$$* M\left(y_{n+3}, y_{n+p}, (k-2)t\right)$$

$$\geq M\left(y_0, y_1, \tfrac{(1-k)t}{k^n}\right) * M\left(y_0, y_1, \tfrac{t}{k^n}\right) * M\left(y_0, y_1, \tfrac{(1-k)t}{k^{n+2}}\right)$$

$$* \dots * M\left(y_0, y_1, \tfrac{(k-p)t}{k^{n+p+1}}\right)$$

And

$$N\left(y_n, y_{n+p}, t\right) \leq N(y_n, y_{n+1}, (1-k)t) \diamond N\left(y_{n+1}, y_{n+p}, kt\right)$$

$$\leq N\left(y_0, y_1, \frac{(1-k)t}{k^n}\right) \diamond N\left(y_{n+1}, y_{n+2}, t\right) \diamond N\left(y_{n+2}, y_{n+p}, (k-1)t\right)$$

$$\leq N\left(y_0, y_1, \frac{(1-k)t}{k^n}\right) \diamond N\left(y_0, y_1, \frac{t}{k^n}\right) \diamond N\left(y_{n+2}, y_{n+3}, t\right) \dots\dots\dots\dots$$

$$\diamond N\left(y_{n+3}, y_{n+p}, (k-2)t\right)$$

$$\leq N\left(y_0, y_1, \frac{(1-k)t}{k^n}\right) \diamond N\left(y_0, y_1, \frac{t}{k^n}\right) \diamond N\left(y_0, y_1, \frac{(1-k)t}{k^{n+2}}\right) \dots\dots\dots\dots$$

$$\diamond N\left(y_0, y_1, \frac{(k-p)t}{k^{n+p+1}}\right)$$

Thus $M\left(y_n, y_{n+p}, t\right) \to 1$ *and* $N\left(y_n, y_{n+p}, t\right) \to 0$

(for all $t > 0$ t *as* $n \to \infty$ *uniformly on* $p \in N$*). Therefore* $\{y_n\}$ *is a Cauchy sequence in X.*

But $(X, M, N, *, \diamond)$ *is complete so there exists a point (say)* z *in* X *such that* $\{y_n\} \to z$.

Also, using 2.2.1(I) we have

$$\{Px_{2n-2}\}, \{Tx_{2n-1}\}, \{Sx_{2n}\}, \{Qx_{2n+1}\} \to z.$$

Since the pair (P, S) *is reciprocally continuous mappings, then we have,*

$$\lim_{n\to\infty} PSx_{2n} = Pz \quad and \quad \lim_{n\to\infty} SPx_{2n} = Sz$$

and compatibility of P *and* S *yields,*

$$\lim_{n\to\infty} M(PSx_{2n}, SPx_{2n}, t) = 1 \ and \ \lim_{n\to\infty} N(PSx_{2n}, SPx_{2n}, t) = 0.$$

i.e. $M(Pz, Sz, t) = 1$ *and* $N(Pz, Sz, t) = 0$.

Hence $Pz = Sz$. *Since* $P(X) \subset T(X)$, *then there exists a point* u *in* X *such that* $Pz = Tu$.

Now by contractive condition, we get,

$$M(Pz, Qu, kt) \geq min\left\{\begin{matrix} M(Sz, Tu, t), M(Pz, Sz, t), M(Qu, Tu, t), \\ M(Pz, Tu, t) \end{matrix}\right\}$$

$$\geq min\left\{\begin{matrix} M(P\,z, P\,z, t), M(Pz, Pz, t), M(Qu, Pz, t), \\ M(Pz, Pz, t) \end{matrix}\right\}$$

$$> M(Pz, Qu, t)$$

$$N\,(Pz, Qu, kt) \leq min\begin{Bmatrix} N\,(Sz, Tu, t), N\,(Pz, Sz, t), N(Qu, Tu, t), \\ N\,(Pz, Tu, t) \end{Bmatrix}$$

$$\leq min\begin{Bmatrix} N\,(P\,z, P\,z, t), N\,(Pz, Pz, t), N(Qu, Pz, t), \\ N\,(Pz, Pz, t) \end{Bmatrix}$$

$$<\ N\,(Pz, Qu, t)$$

i.e. $Pz = Qu$. *Thus* $Pz = Sz = Qu = Tu$. *Since P is S - absorbing then for* $R > 0$ *we have,*

$$M\,(Sz, SP\,z, t)\ \geq\ M\left(Sz, P\,z, \frac{t}{R}\right) = 1$$

$$N\,(Sz, SP\,z, t)\ \leq\ N\left(Sz, P\,z, \frac{t}{R}\right) = 0$$

i.e. $P\,z = SPz = Sz$.

Now by contractive condition, we have,

$$M\,(P\,z, PP\,z, t) = M(PPz, Qu, t)$$

$$\geq min\begin{Bmatrix} M(SP\,z, Tu, t), M\,(PPz, Su, t), \\ M\,(Qu, Tu, t), M(PPz, Tu, t) \end{Bmatrix}$$

$$= min\begin{Bmatrix} M(Pz, Pz, t), M\,(PP\,z, Pz\,, t), \\ M\,(Qu, Qu, t), M\,(PP\,z, Pz, t) \end{Bmatrix}$$

$$= M(PPz, Pz, t)$$

$$N\,(P\,z, PP\,z, t) = N(PPz, Qu, t)$$

$$\leq min\left\{\begin{pmatrix} N(SP\,z, Tu, t), N\,(PPz, Su, t), \\ N\,(Qu, Tu, t), N(PPz, Tu, t) \end{pmatrix}\right\}$$

$$= min\left\{\begin{pmatrix} N(Pz, Pz, t), N\,(PP\,z, Pz\,, t), \\ N\,(Qu, Qu, t), N\,(PP\,z, Pz, t) \end{pmatrix}\right\}$$

$$= N(PPz, Pz, t)$$

i.e. $PP\,z = Pz = SPz$. *Therefore* Pz *is a common fixed point of P and S.*

Similarly, T *is* Q *– absorbing therefore we have,*

$$M\,(Tu, TQu, t) \geq M(Tu, Qu, \frac{t}{R}) = 1$$

$$N\,(Tu, TQu, t) \leq N\left(Tu, Qu, \frac{t}{R}\right) = 0$$

i.e. $Tu = TQu = Qu$. *Now by contractive condition, we have*

$$M\,(QQu, Qu, t) = M(Pz, QQu, t)$$

$$\geq min\left\{\begin{pmatrix} M(Sz, TQu, t), M\,(Pz, Su, t), \\ M\,(QQu, TQu, t), M(Pz, TQu, t) \end{pmatrix}\right\}$$

$$= min\left\{\begin{pmatrix} M(Sz, Qu, t), M\,(Pz\,, Pz, t), \\ M\,(QQu, Qu, t), M\,(Pz, Qu, t) \end{pmatrix}\right\}$$

$$= M(QQu, Qu, t)$$

$$N\,(QQu, Qu, t) = N(Pz, QQu, t)$$

$$\leq min\left\{\begin{pmatrix} N(Sz, TQu, t), N(Pz, Su, t), \\ N(QQu, TQu, t), N(Pz, TQu, t) \end{pmatrix}\right\}$$

$$= min\left\{\begin{pmatrix} N(Sz, Qu, t), N\,(Pz\,, Pz, t), \\ N(QQu, Qu, t), N\,(Pz, Qu, t) \end{pmatrix}\right\}$$

$$= N(QQu, Qu, t)$$

i.e. $QQu = Qu = TQu$.

Hence $Qu = Pz$ *is a common fixed point of P, Q, S and T.*

Uniqueness of Pz can easily follows from contractive condition.

The proof is similar when Q and T are assumed compatible and reciprocally continuous.

This completes the proof. Now we prove the result by assuming the range of one of the mappings P, Q, S *or* T *to be a complete subspace of X.*

Theorem 3.2 *Let* P *be point wise* S *- absorbing and* Q *be point wise* T *– absorbing self maps on an intuitionistic fuzzy metric space* $(X, M, N, *, \diamond)$ *with continuous t-norm defined by a * b = min {a, b} and* $a \diamond b = max\,\{a, b\}$ *where* $a, b \in [0, 1]$ *satisfying the conditions:*

3.2 (I) $P(X) \subseteq T(X), Q(X) \subseteq S(X)$

3.2 (II) There exists $k \in (0, 1)$ *such that for every* $x, y \in X$ *and* $t > 0$

$$M\,(P\,x, Qy, kt) \geq min\left\{\begin{pmatrix} M(Sx, Ty, t), M\,(Px, Sx, t), \\ M\,(Qy, Ty, t), M(P\,x, T\,y, t) \end{pmatrix}\right\}$$

$$N\,(P\,x, Qy, kt) \leq min\left\{\begin{pmatrix} N(Sx, Ty, t), N\,(Px, Sx, t), N\,(Qy, Ty, t), \\ N(P\,x, T\,y, t) \end{pmatrix}\right\}$$

3.2 (III) for all $x, y \in X$,

$\lim_{n\to\infty} M\,(x, y, t) = 0$ *and* $\lim_{n\to\infty} N\,(x, y, t) = 0$. *If the range of one of the mappings maps* P, Q, S *or* T *be a complete subspace of X. Then* P, Q, S *and* T *have a unique common fixed point in X.*

Proof: *let* x_0 *be any arbitrary point in X, construct a sequence* $y_n \in X$ *. Such that*

$y_{2n-1} = Tx_{2n-1} = Px_{2n-2}$ and $y_{2n} = Sx_{2n} = Qx_{2n+1}$, $n = 1, 2, 3,$

This can be done by the virtue of 3.2 (I) and by using the same techniques of above theorem we can show that $\{y_n\}$ *is a Cauchy sequence.*

Let S(X) the range of X be a complete metric subspace than there exists a point such that

$$\lim_{n\to\infty} Sx_{2n} = Su.$$

By3.2 (I) we get

$$Qx_{2n+1} \to Su, Px_{2n-2} \to Su\ , Tx_{2n-1} \to Su\ \text{ and } \{y_n\} \to Su$$

as $n \to \infty$.

By using contractive condition we obtain,

$$M(Px, Qx_{2n+1}, kt) \geq min\left\{\begin{pmatrix} M(Su, Tx_{2n+1}, t), M\ (Pu, Su, t), \\ M(Qx_{2n+1}, Tx_{2n+1}, t), M(Pu, T\ x_{2n+1}, t)\end{pmatrix}\right\}$$

$$N(Px, Qx_{2n+1}, kt) \leq min\left\{\begin{pmatrix} N(Su, Tx_{2n+1}, t), N\ (Pu, Su, t), \\ N(Qx_{2n+1}, Tx_{2n+1}, t), N(Pu, T\ x_{2n+1}, t)\end{pmatrix}\right\}$$

Letting $n \to \infty$*, we get*

$$M(Pu, Su, kt) \geq min\left\{\begin{pmatrix} M(Su, Su, t), M\ (Pu, Su, t), \\ M(Su, Su, t), M(Pu, Su, t)\end{pmatrix}\right\}$$

$$N(Pu, Su, kt) \leq min\left\{\begin{pmatrix} N(Su, Su, t), N\ (Pu, Su, t), \\ N(Su, Su, t), N(Pu, Su, t)\end{pmatrix}\right\}$$

i.e. $Pu = Su$. *Since* $P(X) \subseteq T(X)$, *then there exists* $w \in X$ *such that* $Su = Tw$.

Again by using contractive condition we get,

$$M(Pu, Qw, kt) \geq min\left\{\begin{pmatrix} M(Su, Tw, t), M\ (Pu, Su, t), \\ M(Qw, Tw, t), M(Pu, Tw, t)\end{pmatrix}\right\}$$

$$N(Pu, Qw, kt) \leq min\left\{\begin{pmatrix} N(Su, Tw, t), N(Pu, Su, t), \\ N(Qw, Tw, t), N(Pu, Tw, t)\end{pmatrix}\right\}$$

i.e $Pu = Su = Qw = Tw$. *Since P is pointwise S-absorbing then we have*

$$M(Su, SPu, t) \geq M\left(Su, Qu, \frac{t}{R}\right)$$

$$N(Su, SPu, t) \leq N\left(Su, Qu, \frac{t}{R}\right)$$

i.e. $Su = SPu = SSu$, *and similarly Q is pointwise T -absorbing then we have*

$$M(Tw, TQw, t) \geq M\left(Tw, Qw, \frac{t}{R}\right)$$

$$N(Tw, TQw, t) \leq N\left(Tw, Qw, \frac{t}{R}\right)$$

i.e. $Tw = TQw = QQw$. *Thus* $Su(= Tw)$ *is a common fixed point of* P, Q, S *and* T .

Uniqueness of common fixed point follows from contractive condition.

The proof is similar when $T(X)$*, the range of* T *is assumed to be a complete subspace of X.*

Moreover, Since $P(X) \subseteq T(X)$ *and* $Q(X) \subseteq S(X)$

The proof follows on similar line when either the range of P *or the range of* Q *is assumed complete. This completes the proof of the theorem. Now we give an example to illustrate our theorem.*

Example 3.3 *Let* $X = [2, 20]$ *and* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric. Define mappings* $P, Q, S, T: X \to X$ *by*

$$P(x) = \begin{cases} 2 & if\ x = 2 \\ 3 & if\ x > 2 \end{cases} \qquad S(x) = \begin{cases} 2 & if\ x = 2 \\ 6 & if\ x > 2 \end{cases}$$

$$Q(x) = \begin{cases} 2 & if\ x = 2 \\ 8 & if\ 2 < x \leq 5 \\ 2 & if\ x > 5 \end{cases} \quad and \quad T(x) = \begin{cases} 2 & if\ 2 \leq x \leq 5 \\ x - 3 & if\ x > 5 \end{cases}$$

Also, we Define,

$$M(x, y, t) = \frac{t}{t + |x - y|}, \ N(x, y, t) = \frac{|x - y|}{t + |x - y|}$$

for all $x, y \in X$ *and* $t > 0$.

Then P, Q, S *and* T *satisfy all the conditions of the above theorem with* $k \in (0, 1)$ *and have a unique common fixed point* $x = 2$.

Here, P *and* S *are reciprocally continuous compatible maps. But neither* P *nor* S *is continuous, even at the common fixed point* $x = 2$.

The mapping Q *and* T *are non-compatible but* Q *is pointwise* T *- absorbing. To see* Q *and* T *are non compatible let us consider the sequence* $\{x_n\}$ *in* X *defined by*

$$x_n = 5 + \frac{1}{n}, n \geq 1.$$

Then $\{Tx_n\}, \{Qx_n\}, \{TQx_n\} \to 2$ *and*, $\{QTx_n\} \to 8$ *. Hence B and T are noncompatible.*

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