Fixed Point Theorem for Densifying Mapping in Complete Metric Space

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Abstract: - In this paper, we shall prove a fixed point theorem for continuous densifying mapping.

Keywords: - Complete metric space, Fixed point, Compactness, Densifying mapping

Introduction: -Some fundamental properties about measure of non-compactness α of bounded sets in a metric space are given by Kuratowski [1]Mussbaum [2] Iseki [3] .

Furi and Vignoli (2) have proved the following theorem -

Theorem 1 :-Let T be a continuous, densifying mapping of a bounded complete metric space (X,d) into itself. If for every x, y inX, $x \neq y$ in x , $d(Tx, Ty) < d(x, y)$ then T has a fixed point.

Afterwards Iseki (3) generalized the above result and proved the following theorem :-

Theorem 2 :- Let T be a continuous, densifying mapping of a bounded complete metric space (X,d) itself. If for every x, y in X, $x \neq y$, $x \neq Tx$, $d(Tx, Ty) < ad(x, y) + b\{d(x, Tx) + d(y, Ty)\}$ where a,b are non- negative and a+2b=1 then T has a fixed point.

Definition :- Let (X,d) be a metric space. T be a mapping of X into itself. The mapping T is called densifying if for every bounded sub set A of X with $\alpha(A) > 0$ we have $\alpha(T(A)) < \alpha(A)$.

In this paper we shall prove a fixed point theorems for two continuous densifying mapping.

Our Main Result

Theorem:-Let S and T be a two continuous densifying mapping of a bounded complete metric space (X,d) satisfying conditions

[I]For every x,y in X, $x \neq y$, $x \neq Ty$

$$[d(Sx, STy)] < \alpha\left[\frac{d(Ty,Sx)\sqrt{d(x,STy)}+d(x,Sx)\sqrt{d(Ty,STy)}}{d(x,Ty)}\right]^2$$

$$+\beta\left[\frac{d(x,Sx)\sqrt{d(Ty,STy)}+d(x,STy)\sqrt{d(Ty,Sx)}}{d(x,Ty)}\right]^2 + \gamma\left[\frac{\sqrt{d(x,Ty)d(x,STy)}+\sqrt{d(x,Ty)d(Ty,Sx)}}{\sqrt{d(x,STy)+d(Ty,Sx)}}\right]^2$$

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[II]ST=TS

Where α, β and γ are non-negative reals and $\alpha + \beta + \gamma = 1$ then S and T have a common fixed point in X which is unique if $(\alpha + \beta + 2\gamma) = 1$.

Proof :-Let x_0 be a point of X and we define sequence$\{x_n\}$ *such that* $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$ in Xfor n= 0,1,2,3,…….Put A= { x_{2n+1}:n=0,1,2,3..}Then ST (A) $\subset$ A and by the continuity of S and T. We have S $T(\overline{A}) \subset S\overline{T(A)} \subset \overline{A}$ Hence $(\overline{A})$ isinvariant under S ,T and is bounded.Suppose $\alpha(A) > 0$, since $A = ST(A) \cup$ $\{x_i\}$ We have

$$\alpha(A) = Max\{\alpha(ST(A)), \alpha(x_0)\} = \alpha(ST(A)) < \alpha(A)$$

This is a contradiction.Since the mappings S and T is densifying so $\alpha(A) = 0$ which implies that A is pre compact and since X is complete metric space. $\overline{A}$ is compact define a real valued function f on X by f(x)= d(Tx,STx). By the hypothesis, d(x,Tx) is continuous on the compact subject $\overline{A}$, Hence d(x,Tx) has a minimum point u in $\overline{A}$. To prove that uis a fixed point of S.Suppose $u \neq Su$ we have.

$$f(Su)= [d(STu,STSu)] < \alpha\left[\frac{d(TSu,STu)\sqrt{d(Tu,STSu)+d(Tu,STu)}\sqrt{d(TSu,STSu)}}{d(Tu,TSu)}\right]^2$$

$$+\beta\left[\frac{d(Tu,STu)\sqrt{d(TSu,STSu)+d(Tu,STSu)}\sqrt{d(TSu,STu)}}{d(Tu,TSu)}\right]^2$$

$$+\gamma\left[\frac{\sqrt{d(Tu,TSu)d(Tu,STSu)}+\sqrt{d(Tu,TSu)d(TSu,STu)}}{\sqrt{d(Tu,STSu)+d(TSu,STu)}}\right]^2$$

$$(1-\alpha-\beta)[d(STu,STSu)] < \gamma\ d(Tu,TSu)$$

$$[d(STu,STSu)] < [\frac{\gamma}{(1-\alpha-\beta)}]\ d(Tu,TSu)$$

i.e. $[d(STu,STSu)] < d(Tu,TSu)$

This is a contradiction.So we haveu is a fixed point of S i.e. Su=u. We have $STu = TSu = Tu$.

Now we shall prove that Tu=u .If not, let us suppose that Tu≠u then by [1] we have $d(u,Tu) = d(Su,STu) <$

$$\alpha\left[\frac{d(Tu,u)\sqrt{d(u,Tu)+d(u,u)}\sqrt{d(Tu,Tu)}}{d(u,Tu)}\right]^2$$

$$+\beta\left[\frac{d(u,u)\sqrt{d(Tu,Tu)+d(u,Tu)}\sqrt{d(Tu,u)}}{d(u,Tu)}\right]^2 + \gamma\left[\frac{\sqrt{d(u,Tu)d(u,Tu)}+\sqrt{d(u,Tu)d(Tu,u)}}{\sqrt{d(u,Tu)+d(Tu,u)}}\right]^2$$

$$< \alpha d(Tu,u) + \beta d(Tu,u) + 2\gamma\, d(Tu,u)$$

$$< (\alpha + \beta + 2\gamma)\, d(Tu,u)$$

i.e.$d(u,Tu) < d(Tu,u)$

This is contradiction .So Tu=u.

Uniqueness:- If we possible let w be the another fixed point of Tsuch that u≠w then $d(u,w) = [d(Su,STw)]$

$$< \alpha\left[\frac{d(w,u)\sqrt{d(u,w)+d(u,u)}\sqrt{d(w,w)}}{d(u,w)}\right]^2$$

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$$+\beta\left[\frac{d(u,u)\sqrt{d(w,w)+d(u,w)}\sqrt{d(w,u)}}{d(u,w)}\right]^2 + \gamma\left[\frac{\sqrt{d(u,w)d(u,w)}+\sqrt{d(u,w)d(w,u)}}{\sqrt{d(u,w)+d(w,u)}}\right]^2$$

$< (\alpha+\beta+2\gamma)d(w,u)$ i.e. $d(u,w) < d(u,w)$

This is contradiction .So u=w.Therefore u is a unique common fixed point This completes the proof.

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