OSCILLATION OF SOLUTION OF TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

Subhashini.J[#1],

M.sc., (M.Phil), Department of Mathematics,Vivekanandha College of Arts and Sciences for Women [Autonomous], Tiruchengode, Namakkal.

Karthikeyan. N[#2],

Faculty of Mathematics, Vivekanandha College of Arts and Sciences for Women [Autonomous], Tiruchengode, Namakkal.

Abstract

We concerned a time fractional partial differential equation with the Neumann boundary condition. We establish some sufficient conditions for oscillation of solution of such equation by using a generalized Riccati technique and the integral averaging method. The main results are illustrated.

Keywords:

Fractional partial differential equation, Oscillation, Fractional integral, fractional derivative.

Introduction

In this paper, we concerned the time fractional partial differential equation of the form

$$\frac{\partial}{\partial t}\left(r(t)D_{+,t}^{\gamma}u(x,t)\right) + q(x,t)f\left(\int_0^t (-v)^{-\gamma}u(x,v)dv\right) = a(t)\Delta u(x,t),$$

$$(x,t) \in H = \Omega \times \mathbb{R}_+ \qquad (1)$$

With the Neumann boundary condition

$$\frac{\partial u(x,t)}{\partial N} = 0,$$

$$(x,t) \in \partial\Omega \times \mathbb{R}_+, \qquad (2)$$

Where $D_{+,t}^{\gamma}u$ is the Riemann- Liouville fractional derivative of order γ of u with respect to t, $\gamma \epsilon\ (0,1)$ is a constant, Δ is the Laplacian operator, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$ and N is the unit exterior normal vector to $\partial\Omega$.

The following conditions are assumed to hold:

(i) $r(t) \in C^1([0,\infty);[0,\infty)), a \in C([0,\infty);\mathbb{R}_+)$;

(ii) $q(x,t) \in C(\bar{H};[0,\infty))$ and $min_{x\in\Omega}q(x,t) = Q(t)$;

(iii)$f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(u)/u \geq \mu$ for certain constant $\mu > 0$ and for all $u \neq 0$.

By a solution of equation (1) we mean a function

$u(x,t)\ C^{1+\gamma}(\bar{\Omega}[0,\infty))$such that

$\int_0^t (t-v)^{-\gamma}u(x,v)dv \in C^1(\bar{H};\mathbb{R}), D_{+,t}^{\gamma}u(x,t) \in C^1(\bar{H};\mathbb{R})$ and satisfies (1) on $\bar{H}$.

In equation (1), solution u is oscillatory in H if it is neither eventually. Positive nor eventually negative, otherwise it is nonoscillatory. Equation (1) is oscillatory if all its solutions are oscillatory.

Preliminaries

In this paper, we use several kinds of definitions of fractional derivatives and integrals such as the Riemann – Liouville left – sided definition on the half-axis $\mathbb{R}_+$. Our convenience, we use the following notation.

$$v(t) = \int_\Omega u(x,t)dx.$$

Definition2.1:

The Riemann– Liouville fractional partial derivative of Order $0 < \gamma < 1$ with respect to t of a function $u(x,t)$ is given by

$$\left(D_{+,t}^{\gamma}u\right)(x,t) :=$$

$$\frac{\partial}{\partial t}\frac{1}{\Gamma(1-\gamma)}\int_0^t (t-v)^{-\gamma}u(x,v)\,dv \qquad (3)$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$ where Γ is the gamma function.

Definition 2.2:

The Riemann-Liouville fractional integral of order $\gamma > 0$ of a function $z: \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$(I_+^{\gamma}\, u)(t) \coloneqq \frac{1}{\Gamma(\gamma)}\int_0^t (-v)^{\gamma-1} z(v)\, dv$$

for $t > 0$ (4)

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Definition 2.3:

The Riemann-Liouville fractional derivative of order $\gamma > 0$ of a function $z\colon \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$(D_+^{\gamma} z)(t) \coloneqq \frac{d^{\lceil\gamma\rceil}}{dt^{\lceil\gamma\rceil}}\left(I_+^{\lceil\gamma\rceil-\gamma} y \right)(t) = \frac{1}{\Gamma(\lceil\gamma\rceil-\gamma)}\frac{d^{\lceil\gamma\rceil}}{dt^{\lceil\gamma\rceil}}\int_0^t (t-v)^{\lceil\gamma\rceil-\gamma-1}\, y(v)dv,$$

for t$> o$ (5)

provided the right hand side is pointwise defined on $\mathbb{R}_+$ where $\lceil\gamma\rceil$ is the ceiling function of γ.

Lemma :

Let y be a solution of $\frac{\partial}{\partial t}\left(r(t)D_{+,t}^{\gamma}u(x,t)\right) + q(x,t)$ $f\left(\int_0^t (t-v)^{-\gamma}u(x,v)dv\right) = a(t)\Delta u(x,t)$

$(x,t) \in H = \Omega \times \mathbb{R}_+$ and

$$H(t) \coloneqq \int_0^t (t-v)^{-\gamma}\, z(v)dv$$

for $\alpha \in (0,1)$ and $t > 0$ (6)

Then

$$H'(t) = \Gamma(1-\gamma)\left(D_+^{\gamma} z\right)(t). \tag{7}$$

Proof

Multiplying and dividing by $\Gamma(1-\gamma)$ in equation (6)

$$H(t) = \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma)}\int_0^t (t-v)^{-\gamma}\; Z(v)dv$$

Differentiating the above equation

$$\frac{d}{dt}H(t) = \frac{d}{dt}\left[\frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma)}\int_0^t (t-v)^{-\gamma}\; \mathrm{Z}(v)\mathrm{d}v\right]$$

$$H'(t) = \Gamma(-\gamma)\left[\frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_0^t (t-v)^{-\gamma}\, Z(v)dv\right]$$

By using (5) in above equation

$$H'(t) = \Gamma(1-\gamma)\left[\frac{1}{\Gamma\lceil\gamma\rceil-\gamma}\frac{d^{\lceil\gamma\rceil}}{dt^{\lceil\gamma\rceil}}\int_0^t (t - v)^{\lceil\gamma\rceil-\gamma-1}\, z(v)dv\right]$$

$$= \Gamma\,(1-\gamma)(D_+^{\gamma} z)(t)$$

Hence the proof

Theorem

If the fractional differential inequality

$$\frac{d}{dt}\left[r(t)D_+^{\gamma}\, v(t)\right] + Q(t)f(H(t)) \le 0 \tag{8}$$

has no eventually positive solution, then every solution of (1) and (2)is oscillatory in H.

Proof

Suppose that u is a non oscillatory solution of (1) and (2) is oscillatory in H.

We may assume that u(x ,t) >0 in H $\times[t_0,\infty)$ for some $t_0 > 0$.

Integrating (1) over Ω, we obtain

$$\frac{d}{dt}\left[r(t)\left(\int_{\Omega} D_{+,t}^{\gamma}u)(x,t)dx\right)\right] + \int_{\Omega} q(x,t)f\left(\int_0^t (t - v)^{-\gamma}\, u(x,v)dv\right)dx$$

$$= a(t)\int_{\Omega}\Delta u(x,t)dx. \tag{9}$$

Using Green's formula, it is obvious that

$$\int_{\Omega}\Delta u(x,t)dx \le 0,\ t \ge t_1 \tag{10}$$

By using Jensen's inequality and (I), we have

$$\int_{\Omega} q(x,t)f\left(\int_0^t (t-v)^{-\gamma}u(x,v)dv\right)dx$$

$$\ge Q(t)\left(\int_{\Omega}\left(\int_0^t (t-v)^{-\gamma}u(x,v)dv\right)dx\right)$$

$$= Q(t)f\left(\int_0^t (t-v)^{-\gamma}\left(\int_{\Omega} u(x,v)dx\right)dv\right)$$

(11)

Combining equation (9) - (11) and by using definition, we have

$$\frac{d}{dt}\left[r(t)D_+^{\gamma}v(t)\right] + Q(t)f\big(H(t)\big) \le 0 \tag{12}$$

Since,$H(t) = \left(\int_0^t (t-v)^{-\gamma}\left(\int_{\Omega} u(x,v)dx\right)dv\right)$

Therefore $v(t)$ is an eventually positive solution of (8). This contradicts the hypothesis and completes the proof.

Theorem

Suppose that the conditions (I)-(II)and

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\, dt = \infty \qquad (13)$$

Furthermore, assume that there exists a positive function $c \in C^1[t_0, \infty)$ such that

$$\lim_{t\to\infty} sup \int_{t_1}^{t} \left[\mu c(s)Q(s) - \frac{1}{4}\frac{\left(c'(s)\right)^2 r(s)}{c(s)\Gamma(1-\gamma)}\right] ds = \infty$$

(14)

Then every solution of (8) is oscillatory

Proof

Suppose that $v(t)$ is a non oscilatory solution of (8).

Without loss of generality we may assume that v is an eventually positive solution of (8).

Then there exists $t_1 \geq t_0$, Such that $v(t) > 0$ for $t \geq t_1$.Then it is obvious that

$$\left[r(t)D_+^\gamma(v(t))\right]' \leq -Q(t)f\left(H(t)\right) < 0\,, t \geq t_0.$$

(15)

Thus

$D_+^\gamma v(t) \geq 0$ or $D_+^\gamma v(t) < 0, t \geq t_1$ for some $t_1 \geq t_0$.

We now claim that

$\left(D_+^\gamma v(t)\right) \geq 0$ for $t \geq t_1$.

Suppose not, then $\left(D_+^\gamma v(t)\right) < 0$ and there exists $T \geq t_1$such that $\left(D_+^\gamma v(t)\right) < 0$.

Since $\left[r(t)\left(D_+^\gamma v(t)\right)\right]' < 0$ for $t \geq t_1$, It is clear that

$r(t)\left(D_+^\gamma v(t)\right) < r(T)\left(D_+^\gamma v(T)\right)$ for $t \geq T$

Therefore, from equation (6). We have

$$\frac{H'(t)}{\Gamma(1-\gamma)} = \left(D_+^\gamma v(t)\right) \leq \frac{r(T)\left(D_+^\gamma v(T)\right)}{r(t)}.$$

Integrating the above inequality from T to t, we have

$$\frac{H(t)-H(T)}{\Gamma(1-\gamma)} = r(T)\left(D_+^\gamma v(t)\right)\int_T^t \frac{1}{r(s)} ds$$

$H(t) = H(T) - \Gamma(1-\gamma)r(T)\left(D_+^\gamma v(T)\right)\int_T^t \frac{1}{r(s)} ds$ Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) \leq -\infty$ which is contradiction.

Hence $\left(D_+^\gamma v(t)\right) \geq 0$ for $t \geq t_1$ holds.

Define the function $\omega$by the generalized Riccati substitution

$$\omega(t) = c(t)\frac{r(t)\left(D_+^\gamma v(t)\right)}{H(t)}$$

for $t \geq t_1$ …(16)

Then we have $\omega(t) > 0$ for $t \geq t_1$. From (III), (6),(8),

It follows that

$$\omega'(t) = c'(t)\left[\frac{r(t)\left(D_+^\gamma v(t)\right)}{H(t)}\right] + c(t)\left[\frac{\left(r(t)(D_+^\gamma v(t))\right)'}{H(t)} - \frac{H'(t)r(t)\left(D_+^\gamma v(t)\right)}{H^2(t)}\right]$$

$$\leq \frac{C'(t)}{C(t)}\omega(t) - \frac{C(t)Q(t)f(H(t))}{H(t)} - \frac{C(t)r(t)\Gamma(1-\gamma)(D_+^\gamma v(t))^2}{H^2(t)}$$

$$\leq \frac{c'(t)}{c(t)}\omega(t) - \mu c(t)Q(t) - \frac{\Gamma(1-\gamma)}{c(t)r(t)}\omega^2(t) \qquad \dots(17)$$

$$= -\mu c(t)Q(t) - \left(\sqrt{\frac{\Gamma(1-\gamma)}{c(t)r(t)}}\,\omega(t) - \frac{1}{2}\sqrt{\frac{c(t)r(t)}{\Gamma(1-\gamma)}}\frac{c'(t)}{c(t)}\right)^2 + \frac{1}{4}\frac{(c'(t))^2 r(t)}{c(t)\Gamma(1-\gamma)}$$

$$\leq -\mu c(t)Q(t) + \frac{1}{4}\frac{(c'(t))'r(t)}{c(t)\Gamma(1-\gamma)}.$$

Integrating both sides from t_1 to t, we have

$$\omega(t) \leq \omega(t_1) - \int_{t_1}^{t}\left[\mu c(s)Q(s) - \frac{1}{4}\frac{(c'(s))^2 r(s)}{c(s)\Gamma(1-\gamma)}\right] ds \qquad \dots(18)$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} \omega(t) \leq -\infty$,which contradicts (14) .

Hence proved.

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