An Application to System of Differential Equations in Military Analysis

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Abstract— In this paper, we determine a way to model combat field losses using system of differential equations. The mathematical models are based on DEs, particularly system of ODEs and its main scope way to predict the outcome of army encounters given some basic information about the forces status. The case of aimed fire combat models using Lanchester's square law.

Keywords: System of Differential equations, Lanchester's square law.

1 Introduction:

In 1916, Ferderick William Lanchester pointed out the significance of the concentration of troops in modern battle. He proposed differential equations from which the expected results would be obtained. Deterministic equations of armed fight are a simple benefit between two opposing forces. In particularly, if one country increases its armaments, another country will do same. Sequentially, the first country react by storing more armaments. This kind of arms race represent as a system of differential equations. The case of un aimed fire using Lanchester's linear law and the case of aimed fire using Lanchester's square law[3]. When present the target, the soldier take the armament then quickly fires the single shot is called as aimed fire. Consider the fire of P troops, to each of whom are using aimed fire to participate some number of Q troops. Each individual Q army have some firing rate α, and his armament has some lethality, $f_{kill\backslash hit}$, and some probability of hit, f_{hit}.We roll these up into a single parameter β by $\beta = \alpha f_{kill\backslash hit} f_{hit}$[5].

2 The case of two Armies:

If two armies battle, with $p(t)$ troops on one side and $q(t)$ on the other side, the rate at which soldiers in one army are put out of action is proportional to the troop strength of their enemy. This rise to the system of differential equations

$$\begin{cases} p' = -Aq(t), p(0) = p_0 \\ q' = -Bp(t), q(0) = q_0 \end{cases}$$

where $A > 0$ and $B > 0$ are constants (called their battling effectiveness coefficients) and p_0 and q_0 are their beginning army strengths.

2.1 Example:

A battle is modeled by

$$\begin{cases} p' = -4q, \ p(0) = 150 \\ q' = -p, \ q(0) = 90 \end{cases}$$

(a) Write the solutions of equation in parametric form

(b) Who wins the battle? when? and also state the losses for each side.

Soln:

Taking Laplace transforms on given system of equations

$$\mathcal{L}[p'(t)] = -4\mathcal{L}[q(t)]$$

$$s\mathcal{L}[p(t)] - p(0) + 4\mathcal{L}[q(t)] = 0$$

$$s\mathcal{L}[p(t)] + 4\mathcal{L}[q(t) = 150 \quad (1)$$

Similarly

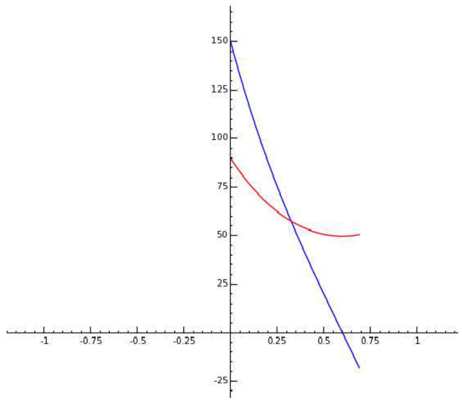

Fig. 1. Lanchester's model for p vs. q combat

$$\mathcal{L}[p(t)] + s\mathcal{L}[q(t)] = 90 \quad (2)$$

Solving equation (1) and (2) we get

$$\mathcal{L}[p(t)] = \frac{150s - 360}{s^2 - 4} \text{ and}$$
$$\mathcal{L}[q(t)] = \frac{90s - 150}{s^2 - 4}$$

Taking inverse Laplace transform on both sides,

$$p(t) = \mathcal{L}^{-1}\left[\frac{150s - 360}{(s-2)(s+2)}\right] \text{and}$$

$$q(t) = \mathcal{L}^{-1}\left[\frac{90s - 150}{(s-2)(s+2)}\right]$$

We get $p(t) = -15e^{2t} + 165e^{-2t}$ and $q(t) = 90cosh(2t) - 75sinh(2t)$

Solving the above equations $p(t) = 0$ is given by $x(t_{win}) = log(11)/4 = 0.59947381...$, and $q(t) = 0$ given by $y(t_{win}) = 49.7493718$, so the number of survivors is 49.

3 Remark

These is a more naive solution to this system which is possible differentiate the first differential equation $p'' = -Aq'$ and then second differential equation into it, $p'' = -A(-Bp) = ABp$. Which implies $p'' = -ABp = 0, p(0) = p_0,\ p'(0) = -Aq(0) = -Aq_0$

The above equation solved by using the method of characteristic polynomials.The characterstic polynomial is the polynomial left-hand side of characteristic equation $\det(A - \lambda I)$, where A is a square matrix and I is square matrix. If you know $p = p(t)$,then solve for q using $q(t) = -\frac{1}{A}p'(t)$.

4 Lanchester's square law:

Suppose q as a function of p instead of p and q as a function of t. One can use the chain rule to derive from the system $p'(t) = -Aq(t), q'(t) = -Bp(t)$ the single equation $\dfrac{dq}{dp} = \dfrac{Bp}{Aq}$

The above equation can be solve by using the method of separation of variables:$Aqdq = Bpdp$, so $Aq^2 = Bp^2 + C$, where C is an unknown constant.(To find C you give initial conditions) The quantity Bp^2 is called by battling strength of the P-men and the quantity Aq^2 is called by battling strength of the Q-men("battling strength" is not to be confused with "troop strength"). The quantity A is called the battling effectiveness of the P-men and the quantity B is called the battling effectiveness of the Q-men.This relationship between troop strengths is called Lanchester's square law and is sometimes expressed as the relative battling strength is a constant: $Aq^2 - Bp^2$=constant

Suppose the total number of troops is some number N, where $p(0)$ are initially placed in a battling capacity and $N - p(0)$ are in a support role. It is often assumed the battling effectiveness is proportional some fixed power of $(N - p(0))/p(0)$(for example, $B = B_0.\dfrac{N - p(0)}{p(0)}$, for a fixed $B_0 > 0$)

If your troops outnumber the enemy then you want to choose the number of support units to be the smallest number such that the battling effectiveness is not decreasing.The remainder should be engaged with the enemy in combat.

5 The case of three Armies:

A combat between three forces gives rise to system of differential equations

$$\begin{cases} p'(t) = -A_1q(t) - A_2r(t),\ p(0) = p_0 \\ q'(t) = -B_1p(t) - B_2r(t),\ q(0) = q_0 \\ r'(t) = -C_1p(t) - C_2q(t),\ r(0) = r_0 \end{cases}$$

where $A_i > 0, B_i > 0$ and $C_i > 0$ are constants and p_0, q_0 and r_0 are the initial troop strengths.

5.1 Example:

The combat modeled by

$$\begin{cases} p'(t) = -q(t) - r(t),\ p(0) = 100 \\ q'(t) = -2p(t) - 3r(t),\ q(0) = 100 \\ r'(t) = -2p(t) - 3q(t),\ r(0) = 100 \end{cases}$$

The $q-$men and $r-$men are better than the $p-$men,in the sense that the coefficient of q in second differential equation is higher than that coefficient of p, and the coefficient of q in third differential equation is also higher then that coefficient of p. We will see, the worst fighter wins.

Taking Laplace transforms,we get

$$\begin{cases} sP(s) + Q(s) + R(s) = 100 \\ 2P(s) + sQ(s) + 3R(s) = 100 \\ 2P(s) + 3Q(s) + sR(s) = 100, \end{cases}$$

Solve by row reduction using the augmented matrix

$$\begin{bmatrix} S & 1 & 1 & 100 \\ 2 & s & 3 & 100 \\ 2 & 3 & s & 100 \end{bmatrix}$$

This have row-reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & (100s + 100)/(s^2 + 3s - 4) \\ 0 & 1 & 0 & (100s - 200)/(s^2 + 3s - 4) \\ 0 & 0 & 1 & (100s - 200)/(s^2 + 3s - 4) \end{bmatrix}$$

This means,

$$P(s) = \frac{100s + 100}{s^2 + 3s - 4} \text{ and}$$

$$Q(s) = R(s) = \frac{100s - 200}{s^2 + 3s - 4}$$

Take inverse Laplace transforms, we get

$$p(t) = 40e^t + 60e^{-4t} \text{ and}$$

$$q(t) = r(t) = -20e^t + 120e^{-4t}.$$

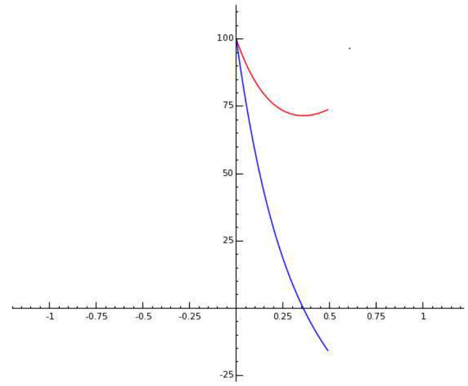

Fig. 2. Lanchester's model for p vs. q vs r combat

In fact, the combat is over at $t = log(6)/5 = 0.35...$ and the time $p(t) = 71.54..$ and $q(t) = r(t) = 1.21..$ Therefore we get, the worst fighters, the $P-$men, not only own but these have lost less than 30% of their men.

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