ON CHARACTERIZATION OF k-J-EP MATRICES

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Abstract

The Concept of k - J - EP matrices is introduced. Relation between k - EP and EP matrices are discussed. As a generalization of k -Hermitian, k - EP matrices and J - EP matrices. Necessary and Sufficient Condition are determined for a matrix to be k - J - EP_r (k - EP , J - EP and rank r). Equivalent Characterization of k - J - EP matrix are discussed. As an application, we shown that the class of all J - EP and k - EP having the same range space form a group under multiplication.

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Introduction

In matrix theory, we come across many special types of matrices and one among them is the normal matrix, which plays an important role in the spectral theory of rectangular matrices. In 1918, the concept of a normal matrix with entries from the complex field was introduced by O.Toeplitz [11] who gave a necessary and sufficient condition for a complex matrix to be normal [1,3,4]. As a generalization of normality the concept of EP matrices over the complex field was introduced by Schwerdtfeger [10]. The class of complex EP matrices includes the class of all non singular matrices, Hermitian matrices and normal matrices. The concept of k - EP [7] matrix is introduced for complex matrices and exhibited as a generalization of k -Hermitian and EP matrices.

An indefinite inner product in C^n is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0, \forall \, y \in C^n$ only when $x = 0$. Associated with any indefinite inner product is a unique invertible Hermitian matrix J (called a weight with Complex entries such that $[x, y] = < x, Jy >$. Where $<,>$ denotes the Euclidean inner product of C^n and vice versa. Motivated by the notion of Minkowski space (as studied by physcists), we also make an additional assumption on namely $J^2 = I$ [9]. Investigation of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors (see for instance [2,8]).

More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices respectively, is denoted to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is defined to be the matrix $J_nA^*J_m$ where J_m and J_n weights in appropriate spaces.

Definition 1.1:

$A \in C_{n \times n}$ is said to be k - J - EP if it satisfies the condition equivalently $N(A) = N(KA^{[*]}K) = N(KJA^*JK)$. A is said to be k - J - EP_r, if A is k - J - EP and of rank r. where $A^{[*]} = JA^*J$.

In particular, when $k(i) = i$ and $j(i) = i$, for each $i = 1$ to n, then the associated permutation matrix K reduces to the identity matrix and definition (1.1) reduces to $N(A) = N(A^*)$ which implies that A is EP matrix [10]. If A is non singular, then A is k - J - EP for all transpositions 'k' in s_n.

Remark 1.2:

We note that a k - J -Hermitian matrix A is k - J - EP . For, if A is k - J -Hermitian, then by known theorem, $A = KA^{[*]}K = KJA^{*}JK$. Hence $N(A) = N(KA^{[*]}K) = N(A^{*}JK)$ which implies A is k - J - EP . However the converse need not be true.

Example 1.3:

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $J = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For a transposition k =(1 2 3) the associated permutation matrix $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $KJA^{*}JK = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq A$. Therefore A is not k - J - Hermitian. A being a non singular matrix, is a k - J - EP matrix. Thus, the class of k - J - EP matrices is a wider class containing the class of k - J - Hermitian matrices.

Theorem 1.4:

For $A \in C_{n \times n}$ the following are equivalent:

(1) A is k - J - EP

(2) KJA is EP

(3) AJK is EP

(4) $A^{\dagger}$ is k - J - EP

(5) $N(A) = N(A^{\dagger}JK)$

(6) $N(A^{*}) = N(AJK)$

(7) $R(A) = R(\mathrm{KJA}^{*})$

(8) $R(A^{*}) = R(\mathrm{KJA})$

(9) $KJA^{\dagger}A = AA^{\dagger}JK$

(10) $A^{\dagger}AJK = KJAA^{\dagger}$

(11) $A = KJA^{*}JKH$ for a nonsingular $n \times n$ matrix H .

(12) $A = HKJA^{*}JK$ for a nonsingular $n \times n$ matrix H .

(13) $A^{*} = HKJAJK$ for a nonsingular $n \times n$ matrix H .

(14) $A^{*} = KJAJKH$ for a nonsingular $n \times n$ matrix H .

(15) $C_n = R(A) \oplus N(AJK)$

(16) $C_n = R(\mathrm{KJA}) \oplus N(A)$

Proof:

The proof for the equivalance of (1), (2) and (3) runs as follows:

A is k - J - EP $\Leftrightarrow$ $N(A)=N(A^{*}JK)$ [By definition (1.1)] $\Leftrightarrow$ $N(\mathrm{KJA})=\mathrm{N}(\mathrm{KJA})^{*}$ $\Leftrightarrow$ KJA is EP $\Leftrightarrow$ $(\mathrm{KJ})(\mathrm{KJA})(\mathrm{KJ})$ is EP $\Leftrightarrow$ AJK is EP. Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold.

(2) $\Leftrightarrow$ (4)

KJA is EP $\Leftrightarrow$ $(\mathrm{KJA})^{\dagger}$ is EP $\Leftrightarrow$ $A^{\dagger}JK$ is EP $\Leftrightarrow$ $A^{\dagger}$ is k - J - EP [By equivalance of (1) and (3) applied to A]. Thus equivalance of (2) and (4) is proved.

(1) $\Leftrightarrow$ (5)

A is k - J - EP $\Leftrightarrow N(A)=N(A^{*}JK) \Leftrightarrow N(A)=N(\mathrm{KJA})^{*} \Leftrightarrow N(A)=N(\mathrm{KJA})^{\dagger}$ $\Leftrightarrow$ $N(A)=N(A^{\dagger}JK)$. Thus , equivalance of (1) and (5) is proved. The other equivalances with (2) can be proved along the same lines and hence the proof is omitted.

Remark 1.5:

In particular, when A is k - J -Hermitian, then theorem (1.4) reduces to result 2.1 of [3]. In result of 2.8 of [3]. It is stated that $A^{\dagger}$ is k - J -Hermitian for a k - J -Hermitian matrix A only when A is normal. We note that, without any condition on A to be normal the result follows:

A is k - J -Hermitian. $\Leftrightarrow A=KJA^{*}JK \Leftrightarrow A^{\dagger}=(KJA^{*}JK)^{\dagger} \Leftrightarrow A^{\dagger}=KJ(A^{*})^{\dagger}JK=KJ(A^{\dagger})^{*}JK \Leftrightarrow$ $A^{\dagger}$ is k - J - Hermitian. When $k(i)=i,\ j(i)=i$, for each $i=1$ to n, then theorem (1.4) reduces to theorem 1 of [4]. Further, the class of complex normal matrices is a subclass of EP matrices. However this is not the case with k - J - EP matrices.

Example 1.6:

Let $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, J=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$ and $A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$

(i) $A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ is EP as well as k - J - EP.

Let $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, J=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$ and $A=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$

(ii) $A=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$ is k - J - EP but not EP.

.Let $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, J=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$ and $A=\begin{pmatrix}1&1&1\\1&0&0\\1&0&0\end{pmatrix}$

(iii) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is EP but not k - J - EP.

Theorem 1.7:

Let $A \in C_{n\times n}$, then any two of the following conditions imply the other one:

(1) A is EP.

(2) A is k - J - EP.

. (3) $R(A) = R(\mathrm{KJA})$.

Proof:

First, we prove that , whenever (1) holds, then (2) and (3) are equivalent. Suppose (1) holds; then by definition of EP matrix, $R(A) = R(A^*)$, now by theorem (1.4), A is k - J - EP $\Leftrightarrow$ $R(A^*) = R(\mathrm{KJA})$. Therefore, A is k - J - EP $\Leftrightarrow$ $R(A) = R(\mathrm{KJA})$. This completes the proof of (1) and (2) $\Longrightarrow$ (3); (1) and (3) $\Longrightarrow$ (2).

Remark 1.8:

For $O \neq E \neq I_n$, $H_k(E)$ is a non abelian group $\Leftrightarrow$ n > 2. For $A \in C_{n\times n}$, there exist unique k - J - Hermitian matrices P and Q such that $A = P + iQ$. where $P = (1/2)(A + KJA^*JK)$ and $\mathrm{Q} = (1/2i)(A - KJA^*JK)$. In the following theorem, an equivalent condition for a matrix to be k - J - EP is obtained in terms of P, the k - J -Hermitian part of A.

Theorem 1.9:

For $A \in C_{n\times n}$, A is k - J - EP $\Leftrightarrow$ $N(A) \subseteq \mathrm{N(P)}$. Where P is the k - J -Hermitian part of A.

Proof:

If A is k - J - EP , then by theorem (1.4), KJA is EP . Since K is non singular, we have $N(A) = N(\mathrm{KJA}) = N((\mathrm{KJA})^*) = N(A^*JK) = N(\mathrm{KJA}^*\,JK).$ Then, for $x \in N(A)$, both $KJAx = 0$ and $KJA^*JKx = 0$ which implies that $Px = (1/2)(A + KJA^*JK)x = 0.$ Thus $N(A) \subseteq \mathrm{N(P)}$.

Conversely, let $N(A) \subseteq \mathrm{N(P)}$ then $Ax = 0$ implies $Px = 0$ hence $Qx = 0$. Therefore, $N(A) \subseteq N(Q)$. Thus, $N(A) \subseteq N(\mathrm{P}) \cap \mathrm{N(Q)}$. Since both P and Q are k - J -Hermitian, by $P = KJP^*JK$ and $Q = KJQ^*JK$. Hence $N(P) = N(KJP^*JK) = N(P^*JK)$ and $N(\mathrm{Q}) = N(\mathrm{KJQ}^*\,JK) = N(\mathrm{Q}^*\,JK)$ .Now $N(A)$ $\subseteq$ $N(P)$ $\cap$ $N(Q)$ $= N(P^*JK) \cap \mathrm{N(Q^*JK)}$ $\subseteq N(P^* - iQ^*)JK.$ Therefore $N(A) \subseteq \mathrm{N(A^*JK)}$ and $rk(A) = rk(A^*JK)$. Hence $N(A) = \mathrm{N(A^*JK)}$. Therefore A is k - J - EP. Hence the theorem.

Next, we derive equivalent conditions for a matrix to be k - J - EP_r. To make the proof simpler, first let us prove certain lemma.

Lemma 1.10:

Let $B \in C_{n\times n}$ be of the form $B = \begin{pmatrix} D & O \\ O & O \end{pmatrix}$ where D is a non singular matrix. Then the following are equivalent:

(1) B is k - J - EP_r

(2) $R(\mathrm{KJB}) = R(B)$.

(3) BB^* is k - J - EP_r

(4) $KJ = \begin{pmatrix} K_1J_1 & O \\ O & K_2J_2 \end{pmatrix}$ where K_1 and K_2 are permutation matrices of order r and (n-r) respectively, and J_1 and J_2 are of order r and (n-r) respectively with $J_i = J_i^* = J_i^{-1}$, $i = 1$ to 2 .

(5) $kJ = k_1J_1k_2J_2$ where k_1 is the product of disjoint transpositions in S_n leaving (r+1, r+2,…,n) fixed and k_2 is the product of disjoint transpositions leaving (1,2,…,r) fixed and J_1 and J_2 of orders r and (n-r) respectively with $J_i = J_i^* = J_i^{-1}$, where $i = 1$ to 2 .

Proof:

(1) $\Leftrightarrow$ (2)

Since B is $J - EP_r$ $\Leftrightarrow$ B is $J - EP$ with $rk(\mathrm{B}) = r$ $\Leftrightarrow$ $R(B) = R(JB^*)$ .Therefore B is k - J - EP $\Leftrightarrow$ $R(B^*) = R(\mathrm{KJB})$. Therefore B is k - J - EP $\Leftrightarrow R(B) = R(KJB)$

(2) $\Leftrightarrow$ (3)

Since $R(KJB) = R(B)$. $R(A) = R(\mathrm{KJA}^*)$ and $rk(\mathrm{B}) = r$ (since A is k - J - EP_r $R(BB^*) = R(KJ(BB^*)^*)$ and $rk(\mathrm{BB}^*) = r$ (since $A = BB^*$). Therefore BB^* is k - J - EP_r .

(2) $\Leftrightarrow$ (4)

$$R(KJB) = R(B) \quad \Leftrightarrow \quad (\mathrm{KJB})(\mathrm{KJB})^\dagger = BB^\dagger \Leftrightarrow KJ(BB)^\dagger \mathrm{JK} = BB^\dagger \Leftrightarrow KJBB^\dagger = BB^\dagger JK$$

Let us partition $K = \begin{pmatrix} K_1 & K_3 \\ K_3^T & K_2 \end{pmatrix}$ where K_1 is $r \times r$, and $J = \begin{pmatrix} J_1 & J_3 \\ J_3^T & J_2 \end{pmatrix}$ where J_1 is $r \times r$.

$$KJBB^\dagger = BB^\dagger JK \Leftrightarrow KJ\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} JK$$

$$\Leftrightarrow \begin{pmatrix} K_1J_1 & O \\ K_3J_3^T & O \end{pmatrix} = \begin{pmatrix} K_1J_1 & K_3J_3 \\ O & O \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} K_1J_1 & O \\ O & K_2J_2 \end{pmatrix} = KJ$$

Thus , equivalence of (2) and (4) holds. The equivalence of (4) and (5) is clear from the definition of 'k - J '.

Remark 1.11:

In Lemma (1.4), the form of B is essential can be seen by the following example:

$B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is not EP . For $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, KJB is EP

Lemma 1.12:

$A \in C_{n\times n}$ is k - J - EP_r $\Leftrightarrow$ There exist unitary matrix U and $r\times r$ non singular matrix F such that

$$A = KJU\begin{pmatrix} F & O \\ O & O \end{pmatrix}U^*.$$

Proof:

Let us assume that A is k - J - EP_r. Then by theorem (1.4), $A^* = HKJAJK$, where H is a non singular matrix. Now, A^*A $=(HKJAJK)\mathrm{A}$ $=H(\mathrm{KJA})(\mathrm{KJA})$ $=H(\mathrm{KJA})^2$ $\Rightarrow rk(\mathrm{H(KJA)}^2)$ $\Rightarrow rk(A^*A)$ $=rk(\mathrm{KJA})^2$. Over the complex field A^*A and A have the same rank. Therefore $rk(\mathrm{KJA})^2 = rk(A^*A) = rk(A) = rk(\mathrm{KJA}) \Rightarrow R(\mathrm{KJA})\bigcap N(\mathrm{KJA}) = \{0\} \Rightarrow R(\mathrm{KJA})\bigcap N(A) = \{0\}$. Thus $C_n = R(\mathrm{KJA}) \oplus N(A)$.

Choose an Orthonormal basis $\{x_1, x_2, x_3.....x_n\}$ of $R(\mathrm{KJA}) = R(A^*)$ extend it to be a basis $\{x_1, x_2, x_3.....x_r, x_{r+1}, ...x_n\}$ of C_n. where $\{x_{r+1}, ...x_n\}$ orthonormal basis of $N(A)$. If (u,v) denotes the usual inner product of C_n and $1 \le r < j \le n$, it follows that $x_i \in R(KJA) = R(A^*)$ $\Rightarrow x_i = A^*y$. Therefore $(x_i, x_j) = (A^*y, x_j) = (y, Ax_j) = 0$ (since $x_j \in N(A)$). Hence $\{x_1, x_2, x_3, ...x_n\}$ is an orthonormal basis of C_n. If we consider KJA as the matrix of a linear transposition relative to any orthonormal basis of C_n then $U^*(\mathrm{KJA})U = \begin{pmatrix} F & O \\ O & O \end{pmatrix}$ Where F is r×r non singular matrix. Implies that $\mathrm{A} = KJU\begin{pmatrix} F & O \\ O & O \end{pmatrix}U^*$.

Conversely, if $\mathrm{A} = KJU\begin{pmatrix} F & O \\ O & O \end{pmatrix}U^*, U^*KJAU = \begin{pmatrix} F & O \\ O & O \end{pmatrix}$. $N(\mathrm{KJA}) = N(\mathrm{KJA})^*$ which implies KJA is EP_r and by theorem (1.4), A is k - J - EP_r.

Theorem 1.13:

$A \in C_{n\times n}$ is k - J - EP_r where $k = k_1k_2$ and $J = \begin{pmatrix} J_1 & O \\ O & J_2 \end{pmatrix}$ order of J_1 is r×r, order of J_2 is (n-r) iff A is unitarily k - J -similar to a diagonal block k - J - EP_r matrix B = $\begin{pmatrix} D & O \\ O & O \end{pmatrix}$ where D is r×r non singular matrix. (det $D = 0$).

Proof:

Since A is k - J - EP_r, by lemma (1.12), there exist unitary matrix U and a r×r non singular matrix F such that $A = (\mathrm{KJUJK})(\mathrm{KJ})\begin{pmatrix} F & O \\ O & O \end{pmatrix}U^*$. Since $k = k_1k_2$ the associated permutation matrix is $K = \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix}$ and $J = \begin{pmatrix} J_1 & O \\ O & J_2 \end{pmatrix}$. Hence $A = (\mathrm{KJUJK})\begin{pmatrix} K_1J_1F & O \\ O & O \end{pmatrix}U^* = KJUJK\begin{pmatrix} D & O \\ O & O \end{pmatrix}\mathrm{U}^*$ where $D = K_1J_1F$. Thus A is unitarily k - J -similar to a diagonal block matrix $\begin{pmatrix} D & O \\ O & O \end{pmatrix}$ where D is r×r non singular, B is k - J - EP_r, follows from lemma (1.10).

Conversely, if $B = \begin{pmatrix} D & O \\ O & O \end{pmatrix}$ with D is r×r non singular is k - J - EP_r, the again by using lemma (1.10), $kJ = k_1J_1k_2J_2$ and $K = \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix}$ and $J = \begin{pmatrix} J_1 & O \\ O & J_2 \end{pmatrix}$. Since A is unitarily k - J -similar to B = $\begin{pmatrix} D & O \\ O & O \end{pmatrix}$ there exists unitary matrix U such that

$A = KJUJK\begin{pmatrix} D & O \\ O & O \end{pmatrix}U^*$. Since B is k-J-EP_r, then by theorem (1.4), $KJB = KJ\begin{pmatrix} D & O \\ O & O \end{pmatrix} = U^*KJAU$ is EP_r, KJA is EP_r. Now A is k-J-EP follows from theorem (1.4) and $rk(\mathrm{A}) = r$. Hence A is k-J-EP_r.

Theorem 1. 14:

If A is k-J-EP_r then (λ, x) is a k-J-eigen value, k-J-eigen vector pair for A $\Leftrightarrow$ $(1/\lambda, \mathrm{kJ}(x))$ is a k-J-eigen value, k-J-eigen vector pair for $A^\dagger$.

Proof:

We assume (λ, x) is a k-J-eigenvalue, k-J-eigen vector pair of A. $\Leftrightarrow Ax = \lambda Kx \Leftrightarrow KAx = K^2\lambda Ix \Leftrightarrow KAx = \lambda Ix$ [since $k^2 = I$] $\Leftrightarrow KAx = \lambda Jx$ $\Leftrightarrow JKAx = \lambda J^2 x$ [since $J^2 = I$] $\Leftrightarrow KJAx = \lambda x$ $\Leftrightarrow (\mathrm{KJA})^\dagger x = (1/\lambda)x$ $\Leftrightarrow A^\dagger(KJ)x = (1/\lambda)KJ(x) \Leftrightarrow (1/\lambda, KJx)$ is a k-J-eigen value, k-J-eigen vector pair for $A^\dagger$. Hence the theorem.

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