SYMMETRIC SOLUTIONS OF NONLINEAR ELLIPTIC NEUMANN BOUNDARY VALUE PROBLEMS

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***Abstract**:* Abstract In this paper we prove the symmetry result for solutions of nonlinear elliptic Neumann boundary value problems by using the Alexandrov's method of reflection and maximum principle, on the unit ball in the n-dimensional Euclidean space with n ≥3.

***Index Terms**:* Maximum principles; symmetry; Narrow region principle; nonlinear elliptic Neumann boundary value problems.

I. INTRODUCTION

. Maximum principle is one of the most used tools in the study of elliptic differential equations. It is generalization of the following well known theorem of the elemental calculus, "If f is a function of class C^2 in [a; b] such that the second derivative is positive on (a; b), then the maximum value of the function f attains at the end point of the interval [a; b]". It can be easily notice that the maximum principle gives information about the global behavior of a function over a domain from the information of the qualitative character in the boundary and without explicit knowledge of the same function. The maximum principle allows us to obtain uniqueness of solution of certain problems with conditions of Dirichlet or Neumann type. Also it help to obtain a priori estimate for solutions. These reasons make interesting the study of the maximum principle on several forms and its eneralizations and Hopf Lemma. For example a geometric version of the maximum principle allows us to compare locally surfaces that coincide at a point. Using the method of reection and a version of maximum principle for thin domains, Beresticky and Nirenberg in [2] made a generalization of symmetry result for the solution of Dirichlet BVP in the paper [8]. The maximum principles are very much useful to obtain the uniqueness of the solutions of certain Dirichlet or Neumann boundary value problems. The maximum principle and the Alexandrov reection principle in [1] have been used to prove symmetry with respect to some point, some plane. It is also useful to prove symmetry of domain and to determine asymptotic-symmetric behavior of the solution of some elliptic problems ([2], [8], [9], [14]). The first person who used this technique was Serrin. He proved that " If u is positive solution of the problem

$$\Delta u = -1 \text{ in } \Omega \text{ with } u = 0;$$
$$\frac{\partial u}{\partial \eta} = constant$$

on $\partial\Omega$ then is ball and u is radially symmetric with respect to the center of Ω ".

Our proof shows that the technique used by Berestycki and Nirenberg [2]; Gidas, Ni and nirenberg [8]; Serrin [14]; Ca_arelli, Gidas and Spruck [3]; Dhaigude and Patil [4],[5] , for the study of symmetry of solutions of the Dirichlet elliptic BVP , can be applied to prove the symmetry of the solutions of the Neuman elliptic BVP, with nonlinear term f(x; u(x)). [13] shows a result of symmetry for a big class of problems with condition of Neumann on the boundary in the one dimensional case. We use the method of reection of Alexandrov Maximum principle is one of the most used tools in the study of elliptic diferential equations. It is generalization of the following well known theorem of the elemental calculus,"If f is a function of class C^2 in [a; b] such that the second derivative is positive on (a; b), then the maximum value of the function f attains at the end point of the interval [a; b]". It can be easily notice that the maximum principle gives information about the global behavior of a function over a domain from the information of the qualitative character in the boundary and without explicit knowledge of the same function. The maximum principle allows us to obtain uniqueness of solution of certain problems with conditions of Dirichlet or Neumann type. Also it help to obtain a priori estimate for solutions. These reasons make interesting the study of the maximum principle on several forms and its eneralizations and Hopf Lemma. For example a geometric version of the maximum principle allows us to compare locally surfaces that coincide at a point. Using the method of reection and a version of maximum principle for thin domains, Beresticky and Nirenberg in [2] made a generalization of symmetry result for the solution of Dirichlet BVP in the paper [8]. The maximum principles are very much useful to obtain the uniqueness of the solutions of certain Dirichlet or Neumann boundary value problems. The maximum principle and the Alexandrov reection principle in [1] have been used to prove symmetry with respect to some point, some plane. It is also useful to prove symmetry of domain and to determine asymptotic-symmetric behavior of the solution of some elliptic problems Our proof shows that the technique used by Berestycki and Nirenberg [2]; Gidas, Ni and nirenberg [8]; Serrin [14]; Ca_arelli, Gidas and Spruck [3]; Dhaigude and Patil [4],[5] , for the study of symmetry of solutions of the Dirichlet elliptic BVP , can be applied to prove the symmetry of the solutions of the Neuman elliptic BVP, with nonlinear term f(x; u(x)). [13] shows a result of symmetry for a big class of problems with condition of Neumann on the boundary in the one dimensional case. We use the method of reflection of Alexandrov.

It is interesting to study the behavior of solutions of the elliptic boundary value problem.

$$\Delta u = f(x; u) \text{ in } B \qquad (1.1)$$
$$\frac{\partial u}{\partial \eta} = g(u) \quad \text{on} \quad \partial B$$

Escobar [6] completely studied the set of conformal metrics G, of the standard metric $\acute{o}_{ij}$ in $\bar{B}$. Furthermore, he proved that solutions of boundary value problems are symmetric with respect to some point, where B is unitary ball in R^n and $\frac{\partial}{\partial \bar{\eta}}$ is outward normal derivative to ∂B, f and g are functions defined in R . For $n \geq 3$, the problem (1.1) appears in the study of conformal deformation of the standard metric over the unitary ball ∂B that have constant scalar curvature in B and mean curvature on ∂B. Also Escobar and Garcia [7] studied the conformal metrics on the ball with zero scalar curvature and prescribed mean curvature on the boundary. Recently, author [11] proved symmetry of solutions of system of ordinary differential equations with Neumann boundary conditions. We prove that the solution of the elliptic boundary value problem with Neumann boundary condition,

$$a\Delta u + \sum_{i=1}^{n} u_i + cu = f(x; u) \qquad \text{in} \quad B$$
$$\frac{\partial u}{\partial \eta} = g(u) \quad \text{on} \quad \partial B$$

are radially symmetric with respect to the origin.

Our proof shows that techniques used in [[2],[8], [9]] for the study of symmetric solutions of the elliptic problem with Dirichlet condition.

$$\Delta u + f(u) = 0 \quad \text{in B}$$
$$u = \emptyset \qquad \text{on } \partial B$$

where ∂B is the closed ball, can be applied in elliptic boundary value problems with Neumann condition.

2 Maximum Principle and Hopf Lemma:

Our result is based on the well known maximum principle and on the Hopf boundary lemma for the differential operator of the form,

$$L(u) \equiv \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_l \, \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

Theorem 2.1 [12]-(Maximum principle): Let u(x_1; x_2; x_3; :::; x_n) satisfy the differential inequality $L(u) \geq 0$ with $c(x) \leq 0$ with L uniformly elliptic in Ω , and the coeffcients of L bounded. If u attains a nonnegative maximum M at an interior point of Ω , then u = M.

we prove that solutions of the elliptic problem with Neumann condition are radially symmetric with respect to the origin.

3 Symmetry result:

Let $B_1(0)$ is unit ball with center at origin and radius 1. Let x = (x_1; x_2; x_3; :::; x_n) $\in B_1(0)$. x' = (- x_1; x_2; x_3; :::; x_n) be the point of reflection of x with respect to the hyperplane $x_1 = 0$. We use the symbol,

$$\Delta \equiv \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2} + \ \dots\dots + \frac{\partial^2}{\partial n^2}$$

Theorem 3.1 Let $u \in C^2(B) \cap C^0(\bar{B})$ be a positive solution of the elliptic problem with Neumann condition,

$$a\Delta u + b\sum_{i-1}^{n} u_i + cu = f(x; u) \qquad \text{in} \quad B_1(0)$$
$$\frac{\partial u}{\partial \bar{\eta}} = g(u) \quad \text{on} \quad \partial B_1(0)$$

where $\alpha: \bar{B} \to R$ is a bounded function and symmetric with respect to the origin. such that a(x) > 0 for all $x \in \overline{B_1}$ (0) and $b : \bar{B} \to R$ is a bounded and odd function and c : $\bar{B} \to R$ is a bounded function and symmetric with respect to the origin such that c(x) ≤ 0 for all $x \in \overline{B_1}$ (0). Let $\bar{\eta}$ denote the outward normal vector to ∂B. Let $f \in C^1 (R \times R^+ , R)$ is such that f(x; t) is strictly increasing in t for all x ∈ $B_1(0)$ and is symmetric to x = 0 for all t ∈ R^+ and g is strictly decreasing. Then u is symmetric with respect to the origin.

Proof 1: Let v(x) = u(x') for x ∈ $\bar{B}$ where x' denotes the reflection of x with respect to the hyperplane $x_1 = 0$. Then

$$a(x)\Delta u(x) + b(x)\sum_{i=1}^{n} u_i(x) + c(x)u(x) = f(x; u(x)) \quad \text{in } B_1(0) \qquad (3.1)$$

$$\frac{\partial u(x)}{\partial \eta} = g(u(x)) \quad \text{on} \quad \partial B_1(0) \qquad (3.2)$$

These equations are also satisfied at x_0.

$$a(x')\Delta u(x') + b(x')\sum_{i=1}^{n} u_i(x') + c(x')u(x') = f(x'; u(x')) \quad \text{in } B_1(0) \qquad (3.3)$$

$$\frac{\partial u(x')}{\partial \eta} = g(u(x')) \quad \text{on} \quad \partial B_1(0) \qquad (3.4)$$

Since v(x) = u(x') for $x \in \bar{B}$ and conditions in the statement of theorem, equation (3.3) can be expressed as

$$a(x)\Delta v(x) + b(x)\sum_{i=1}^{n} v_i(x) + c(x)v(x) = f(x; v(x)) \quad \text{in } B_1(0) \qquad (3.5)$$

$$\frac{\partial v(x)}{\partial \eta} = g(v(x)) \quad \text{on} \quad \partial B_1(0) \qquad (3.6)$$

Let w be the function defined by w(x) = u(x) −v(x) for $x \in \bar{B}$

$$\frac{\partial w}{\partial x_1} = \frac{\partial u}{\partial x_1} - \frac{\partial v}{\partial x_1}$$

$$\frac{\partial w}{\partial x_i} = \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \quad \text{for } i \neq -1$$

and similarly

$$\frac{\partial^2 w}{\partial {x_1}^2} = \frac{\partial^2 u}{\partial {x_1}^2} - \frac{\partial^2 v}{\partial {x_1}^2}$$

$$\frac{\partial^2 w}{\partial {x_1}^2} = \frac{\partial^2 u}{\partial {x_1}^2} - \frac{\partial^2 v}{\partial {x_1}^2} \quad \text{for } i \neq -1$$

$$\therefore \Delta w = \Delta u - \Delta v$$

Subtracting the equations we get,

$$a(x)\Delta w(x) + b(x)\sum_{i=1}^{n} w_i(x) + c(x)w(x) = f(x; v(x)) - f(x; v(x)) \quad \text{in } B_1(0) \qquad (3.5)$$

$$\frac{\partial w(x)}{\partial \eta} = g(v(x)) - g(v(x)) \quad \text{on} \quad \partial B_1(0) \qquad (3.6)$$

Since u is continuous function in $\bar{B}$, w is continuous function in $\bar{B}$ therefore there must exist

$$x_{min} \text{ and } x_{min} \in \bar{B}$$

such that

$$w(x_{min}) = {}^{min}_{\bar{B}} w$$

and

$$w(x_{max}) = {}^{max}_{\bar{B}} w$$

We can conclude that x_{min} and x_{max} cannot be in ∂ B. To prove this contrary suppose that x_{max} $\in \partial$B and satisfies $w(x_{max}) > w(x)$for all x $\in \bar{B}$. Then

$$\frac{\partial w\,(x_{max})}{\partial \bar{\eta}} \geq 0 \text{ and } w(x_{max}) > 0$$

in consequence

$$g(u\, x_{max}) - g(v\, x_{max}) \geq 0$$

and

$$g(u\, x_{max}) \geq g(u(x)) \text{ for all } x \in B$$

Since g is strictly decreasing we have

$$u(x_{max}) \leq v(x_{max})$$

$$u(x_{max}) - v(x_{max}) \leq 0$$

$$w(x_{max}) \leq 0$$

This contradicts to $w(x_{max}) > 0$, therefore x_{max} cannot be in ∂B. Now suppose that x_{min} $\in \partial$B and satisfies

$$w(x_{min}) < w(x) \text{ for all } x \in B$$

Then

$$\frac{\partial w\,(x_{min})}{\partial \bar{\eta}} \leq 0 \text{ and } w(x_{min}) < 0$$

in consequence

$$g(u\, x_{min}) - g(v\, x_{min}) \geq 0$$

and

$$g(u\, x_{min}) \geq g(u(x)) \text{ for all } x \in B$$

Since g is strictly decreasing we have

$$u(x_{min}) \geq v(x_{min})$$

$$u(x_{min}) - v(x_{min}) \geq 0$$

$$w(x_{min}) \geq 0$$

Which contradicts to $w(x_{min}) < 0$, therefore x_{min} cannot be in ∂B..

Hence

$$x_{max}\ , x_{min} \in B.$$

Therefore we have

$$a(x_{min})\Delta w(x_{min}) + b(x_{min})\sum_{i=1}^{n} w_i(x_{min}) + \mathrm{c}(x_{min})\mathrm{v}(x_{min}) \geq 0$$

$$a(x_{max})\Delta w(x_{max}) + b(x_{max})\sum_{i=1}^{n} w_i(x_{max}) + \mathrm{c}(x_{max})\mathrm{v}(x_{max}) \leq 0$$

So u is symmetric with respect to $x_n = 0$. Since B is unit ball we can apply the same argument in all directions. It follows that u is radially symmetric with respect to the origin.

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