OPTIMAL ERROR BOUNDS THROUGH INTERPOLATORY SEXTIC SPLINE

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Abstract. The present problem consists of the existence and uniqueness of sextic C^3-splines interpolating to a given data at the mid-points of each sub interval together with the function value and first derivative at the partition points with suitable end conditions. The estimate calculated can not be improved for a uniform partition. The technique used is that of due to Hall and Meyer.

Mathematics Subject Classification (1991) : (AMS) 41A15, 65D07.

1. INTRODUCTION.

In a paper Subotin [9] has considered the problem of the existence and convergence of even degree splines with equidistant mesh points, which interpolate to given data at the mid-points of the mesh intervals. Some of these results have been treated by I.J. Schoenberg [10], G. Birkhoff and C.deBoor [2, 3] and A. Meir and A. Sharma [8] from various point of view. Garry Howell and A.K. Varma [5] presented the best error bounds for quartic spline interpolation by using the technique devised by Hall and Meyer in [6]. In the present paper we describe the problem as follows :

Let

(1.1) $$\Delta : 0 = x_0 < x_1 < \dots\dots\dots\dots < x_n = 1$$

be any subdivision of [0, 1]. Consider the sextic splines s(x) such that

(1.2)

(i) $s(x) \in C^3 [0, 1]$,

(ii) s(x) is a polynomial of deg. ≤ 6 in each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \dots\dots\dots\dots, n$.

Here we state the following theorems, which we shall prove in the next sections.

Theorem 1. Given arbitrary numbers $f(\alpha_i)$, i = 1, 2,......n; $f(x_i)$, i = 0, 1, ... n; $f'(x_i)$, i = 0, 1, ... n; $f'''(x_i)$, i = 0, n, then there exists a unique sextic spline $s(x) \in C^3[0, 1]$ such that

$$s(\alpha_i) = f(\alpha_i) \text{ with } \alpha_i = \frac{x_{i-1} + x_i}{2} \text{ for } i = 1, 2, \ldots, n;$$

(1.3) $$s(x_i) = f(x_i) \ \& \ s'(x_i) = f'(x_i) \text{ for } i = 0, 1, \ldots, n,$$

together with $s'''(x_0) = f'''(x_0)$, $s'''(x_n) = f'''(x_n)$.

Theorem 2. Let $f \in C^7[0, 1]$ and let s(x) be the unique sextic spline satisfying the conditions of Theorem 1. Then we have

(1.4) $$|s(x) - f(x)| \le \frac{M_0 h^7}{8!} \max_{0 \le x \le 1} |f^{(7)}(x)|,$$

where

(1.5) $$M_0 = \left(\frac{1}{26} + \frac{\sqrt{26}}{4}\right)\left(\frac{1}{2} - \frac{1}{\sqrt{26}}\right)^{1/2} = \max_{0 \le t \le 1} |\phi(t)|,$$

and

(1.6) $$\phi(t) = \frac{t^2 (2t-1)(1-t)^2 (5+8t-8t^2)}{2}$$

2. PRELIMINARIES.

If p(u) is a polynomial of degree 6, $0 \le u \le 1$, it is easily seen that

(2.1) $$p(u) = p(0)\, Q_1(u) + p(½)\, Q_2(u) + p(1)\, Q_3(u) + p'(0)\, Q_4(u)$$
$$+ p'(1)\, Q_5(u) + p'''(0)\, Q_6(u) + p'''(1)\, Q_7(u),$$

where

$$Q_1(u) = (1 - u)^2 (1 - 2u)\left(1 + 4u + \frac{19}{6} u^2 - \frac{16}{3} u^3\right),$$

$$Q_2(u) = \frac{32}{3} u^2 (1-u)^2 (1+2u-2u^2)$$

$$Q_3(u) = \frac{1}{6} u^2 (2u-1)(17+34u-77u^2+32u^3),$$

(2.2) $$Q_4(u) = \frac{1}{12} u (1-u)^2 (1-2u)(12+11u-16u^2),$$

$$Q_5(u) = \frac{1}{12} u^2 (1-u)(1-2u)(7+21u-16u^2),$$

$$Q_6(u) = \frac{1}{144} u^2 (2u-1)(1-u)^2 (5-4u),$$

$$Q_7(u) = \frac{1}{144} u^2 (2u-1)(1-u)^2 (1+4u).$$

3. PROOF OF THEOREM 1.

Let s(x) denote the restriction of p(u), $0 \le u \le 1$, in the subinterval $[x_{i-1}, x_i]$ for i = 1, 2, ..., n, we have

(3.1) $$s(x) = f(x_{i-1}) Q_1(t) + f(\alpha_i) Q_2(t) + f(x_i) Q_3(t) + h_i \{f'(x_{i-1}) Q_4(t) + f'(x_i) Q_5(t)\} + h_i^3 \{s'''(x_{i-1}) Q_6(t) + s'''(x_i) Q_7(t)\},$$

where $h_i = x_i - x_{i-1}$ and $t = \frac{x - x_{i-1}}{h_i}$, $0 \le t \le 1$.

On using (2.1) and the conditions

(3.2) $s'''(0) = f'''(0)$ and $s'''(1) = f'''(1)$,

we see that s(x) given by (3.1) does satisfy (1.3) and that it is sextic in $[x_{i-1}, x_i]$ for i = 1, 2, ..., n. We still need to decide whether it is possible to determine $s'''(x_i)$ (i = 1, 2, ..., n-1) uniquely. For this purpose we use the condition of continuity that

(3.3) $s''(x_i -) = s''(x_i +)$, i = 1, 2, ..., n-1.

This condition with the help of (3.1) reduces to

(3.4) $h_i s'''(x_{i-1}) + 5(h_i + h_{i+1}) s'''(x_i) + h_{i+1} s'''(x_{i+1})$

$$= \frac{408}{h_i^2} f(x_{i-1}) + 1128 \left(\frac{1}{h_i^2} - \frac{1}{h_{i+1}^2} \right) f(x_i) - \frac{408}{h_{i+1}^2} f(x_{i+1})$$

$$+ \frac{84}{h_i} f'(x_{i-1}) - 444 \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right) f'(x_i) + \frac{84}{h_{i+1}} f'(x_{i+1})$$

$$- \frac{1536}{h_i^2} f(\alpha_i) + \frac{1536}{h_{i+1}^2} f(\alpha_{i+1}). \quad (i = 1, 2, ..., n-1)$$

Obviously the above system of equations are strictly tridiagonally dominant. With which $s'''(x_i)$ (i = 1, 2, ..., n-1) can be determined uniquely. This establishes Theorem 1.

4. ESTIMATE.

In order to prove Theorem 2, we require the following.

Lemma. Let $f(x) \in C^7 [0, 1]$ and let $e'''(x_i) = s'''(x_i) - f'''(x_i)$, i = 1, 2, ., n-1, then

(4.1) $$\max_{1 \le i \le n-1} |e'''(x_i)| \le \frac{9h^4}{2\,(7!)} R,$$

where $R = \max_{0 \le x \le 1} |f^{(7)}(x)|$ and $h = \max_{i=1,..,n} h_i$.

Proof. Let j be an index such that

$$\max_{1 \le i \le n-1} |e'''(x_i)| = |e'''(x_j)|.$$

From (3.4) it follows that

(4.2) $$h_j e'''(x_{j-1}) + 5(h_j + h_{j+1}) e'''(x_j) + h_{j+1} e'''(x_{j+1}) = P_0(f),$$

where

(4.3) $$P_0(f) = \frac{408}{h_j^2} f(x_{j-1}) + 1128 \left(\frac{1}{h_j^2} - \frac{1}{h_{j+1}^2} \right) f(x_j) - \frac{408}{h_{j+1}^2} f(x_{j+1})$$

$$+ \frac{84}{h_j} f'(x_{j-1}) - 444 \left(\frac{1}{h_j} + \frac{1}{h_{j+1}} \right) f'(x_j) + \frac{84}{h_{j+1}} f'(x_{j+1})$$

$$- \frac{1536}{h_j^2} f(\alpha_j) + \frac{1536}{h_{j+1}^2} f(\alpha_{j+1}) - h_j f'''(x_{j-1})$$

$- 5(h_j + h_{j+1})\, f'''(x_i) - h_{j+1}\, f'''(x_{j+1})$.

As $P_0(f)$ is a linear functional which is zero for polynomials of deg. ≤ 6, we can apply the Peano Theorem [4, p. 70], to obtain

(4.4) $$P_0(f) = \int_{x_{j-1}}^{x_{j+1}} \frac{f^{(7)}(y)}{6!} P_0\left[(x-y)_+^6\right] dy.$$

From the above, we have

(4.5) $$\left| P_0(f) \right| \le \frac{R}{6!} \int_{x_{j-1}}^{x_{j+1}} \left| P_0\left[(x-y)_+^6\right] \right| dy,$$

where $R = \max_{0 \le x \le 1} \left| f^{(7)}(x) \right|$.

Now, we obtain from (4.3) that for $x_{j-1} \le y \le x_{j+1}$

(4.6) $$P_0\left[(x-y)_+^6\right] = 1128 \left(\frac{1}{h_j^2} - \frac{1}{h_{j+1}^2} \right) (x_j - y)_+^6 - \frac{408}{h_{j+1}^2} (x_{j+1} - y)_+^6$$
$$- 2664 \left(\frac{1}{h_j} + \frac{1}{h_{j+1}} \right) (x_j - y)_+^5 + \frac{504}{h_{j+1}} (x_{j+1} - y)_+^5$$
$$- \frac{1536}{h_j^2} (\alpha_j - y)_+^6 + \frac{1536}{h_{j+1}^2} (\alpha_{j+1} - y)_+^6$$
$$- 600 (h_j + h_{j+1}) (x_j - y)_+^3 - 120\, h_{j+1} (x_{j+1} - y)_+^3,$$

and the expression thus obtained can be made symmetric about x_j. Which implies that $P_0\left[(x-y)_+^6\right]$ is non-negative for $x_{j-1} \le y \le x_{j+1}$.

Therefore, as required by the r.h.s. of (4.5), we have

(4.7) $$\int_{x_{j-1}}^{x_{j+1}} \left| P_0\left[(x-y)_+^6\right] \right| dy \le \frac{18}{7} \left(\mathbf{h}_j^5 + \mathbf{h}_{j+1}^5 \right).$$

From (4.5) and (4.7), we obtain

(4.8) $$\left| P_0(f) \right| = \frac{18}{7!} \left(h_j^5 + h_{j+1}^5 \right) R.$$

From (4.2) and (4.8) it follows that

$$|e'''(x_j)| \le \frac{9(h_j^5 + h_{j+1}^5)}{2(7!)\left(h_j + h_{j+1}\right)} R,$$

as $h_j + h_{j+1} \neq 0$ so the term in the denominator cancells out.

We set $h = \max_{i=1,..,n} h_i$ in the above, we have

(4.9) $$|e'''(x_j)| \le \frac{9h^4}{2(7!)} R,$$

which establishes the lemma.

5. PROOF OF THEOREM 2.

Suppose that $L_i[f, x]$ is the unique sextic agreeing with $f(x_{i-1})$, $f(x_i)$, $f(\alpha_i)$, $f(\alpha_{i+1})$, $f'(x_{i-1})$, $f'(x_i)$, $f'''(x_{i-1})$ and $f'''(x_i)$. Let $f \in C^7[0, 1]$ and let $s(x) \in C^3[0, 1]$ be the unique sextic spline under the conditions of Theorem 1. Then, for $x_{i-1} \le x \le x_i$, we have

(5.1) $$|s(x) - f(x)| \le |s(x) - L_i[f, x]| + |L_i[f, x] - f(x)|.$$

By a well- known theorem of Cauchy [4] we obtain

(5.2) $$|L_i[f, x] - f(x)| \le \frac{h^7}{8!} |t^2(1-t)^2(2t-1)(1+4t-4t^2)| R,$$

where $t = \frac{\mathbf{x} - \mathbf{x}_{i-1}}{h_i}$ and $R = \max_{0 \le x \le 1} |f^{(7)}(x)|$.

Now, from (3.1) we have

$$s(x) - L_i[f, x] = h_i^3 e'''(x_{i-1}) Q_6(t) + h_i^3 e'''(x_i) Q_7(t).$$

We set $h = \max_{i=1,..,n} h_i$ and $|e'''(x_j)| = \max_{i=1,..,n-1} |e'''(x_i)|$,

then we have

(5.3) $$|s(x) - L_i[f, x]| \le h^3 |e'''(x_j)| \{|Q_6(t)| + |Q_7(t)|\}.$$

As $Q_6(t)$ and $Q_7(t)$ both are negative for $0 \le t \le ½$ and both are positive for $½ \le t \le 1$, it follows that

(5.4) $$|Q_6(t)| + |Q_7(t)| = |Q_6(t) + Q_7(t)| = \frac{1}{24} | t^2 (2t-1)(1-t)^2 |.$$

Hence, (5.3) with the help of (4.9) and (5.4) gives

(5.5) $$| s(x) - L_i[f, x] | \le \frac{3h^7}{2(8!)} R | t^2 (2t - 1)(1 - t)^2 |.$$

Now, on using (5.2) and (5.5), the inequality (5.1) gives

(5.6) $$| s(x) - f(x) | \le \frac{h^7}{8!} | \phi(t) | R,$$

where

(5.7) $$| \phi(t) | = \frac{3}{\mathbf{2}} | t^2 (2t-1)(1-t)^2 | + | t^2 (2t-1)(1-t)^2 (1+4t-4t^2) |$$

and

$$\phi(t) = \frac{t^2 (2t-1)(1-t)^2 (5 + 8t - 8t^2)}{2},$$

which establishes Theorem 2.

Remark. It is of interest to note that M_0 in (1.5) can not be improved for a uniform partition. For it, we show that inequality (1.4) is best possible in the limit. Let $f_0(x) = x^7/7!$. Then from the Cauchy formula, we have (i = 1,2...,n)

(5.8) $$\left| L_i\left[\frac{t^7}{7!}, \mathbf{x}\right] - \frac{\mathbf{x}^7}{7!} \right| = \frac{h^7}{8!} | t^2 (1-t)^2 (2t-1)(1 + 4t - 4t^2) |.$$

Again, for uniform spacing between knots, we have

(5.9) $$P_o\left(\frac{\mathbf{x}^7}{7!}\right) = \frac{36\ h^5}{7!} = h\, e'''(x_{i-1}) + 10\, h\, e'''(x_i) + h\, e'''(x_{i+1}).$$

From the above, we have

$$\max_{i=1,..,n} | e'''(x_i) | = \frac{9\ h^4}{2(7!)}.$$

On using (5.3), we have

(5.10) $$|s(x) - L_i[f_0, x]| = \frac{9\ h^7}{2\,(7!)}\,|Q_6(t) + Q_7(t)|$$

$$= \frac{3\ h^7}{2\,(8!)}\,|t^2(2t-1)(1-t)^2|.$$

Now combining (5.8) and (5.10), we have

(5.11) $$|s(x) - f(x)| = \frac{h^7}{8!}\left\{\frac{3}{2}|t^2(2t-1)(1-t)^2| + |t^2(2t-1)(1-t)^2(1+4t-4t^2)|\right\},$$

for $x_{i-1} \le x \le x_i$.

From (5.11) we conclude that (1.4) is best possible for uniform spacing in the limit.

Lastly, we state some theorems of the less smooth class of functions without proof as follows :

Theorem 3. Let $f \in C^3[0, 1]$. Let s(x) be the unique sextic spline under the conditions of Theorem 1. Then we have

(5.12) $$|s(x) - f(x)| \le \frac{h^3}{3!} N_0\, \omega_3(f; h),$$

where $\omega_3(*)$ is the modulus of continuity of $f^{(3)}(x)$,

$$N_0 = \max_{0 \le t \le 1} |c_1(t)| \approx 1.0827,$$

and

$$c_1(t) = \left(8t^6 - \frac{63}{4}t^5 - \frac{5}{8}t^4 + \frac{37}{2}t^3 - \frac{89}{8}t^2 + 1\right).$$

Theorem 4. Let $f \in C^5[0, 1]$. Let s(x) be the unique sextic spline under the conditions of Theorem 1.

Then we have

$$|s(x) - f(x)| \le \frac{h^5}{5!} U_0\, \omega_5(f; h),$$

where $\omega_5(*)$ is the modulus of continuity of $f^{(5)}(x)$,

$$U_0 = \max_{0 \le t \le 1} |c_2(t)| \approx 1.0872,$$

and

$$c_2(t) = \left(10t^6 - \frac{85}{4}t^5 + \frac{25}{8}t^4 + \frac{39}{2}t^3 - \frac{99}{8}t^2 + 1\right).$$

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