A Modified Necessary Instability Criterion of In-viscid Homogeneous Shear Flows

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Abstract: A theorem for instability which incorporate in itself the important role played by the square of wave number is derived here in the context of instability of in-viscid homogeneous parallel shear flows. The criterion thus extends the work of Rayleigh (1880), Fjørtoft (1935) and Banerjee et al. (2000).

Some more results concerning the explicit restriction on wave number of arbitrary unstable neutrally stable perturbation waves are also given which supports and unifies the work of Tollmein (1935) Fredrichs (1942) Howard (1964) & Banerjee et al. (1995).

Index Terms **- Homogeneous Parallel Shear flow, Rayleigh Equation, Stability.**

Introduction:

The basic equation describing small perturbation of homogeneous in-viscid parallel shear flows is defined by Rayleigh equation

$$\left(D^2-\alpha^2\right)\phi-\frac{U''}{U-c}\phi=0 \tag{1.1}$$

together with the boundary conditions

$$\phi(z_1)=0=\phi(z_2), \tag{1.2}$$

where $z\in[z_1,z_2]$ is a real independent variable, $D\equiv\frac{d}{dz}$ (also represented by accent), $U(z)$ the basic velocity profile, α a wave number satisfying $0<\alpha^2<\infty$, $c=c_r+\iota c_i$ complex perturbation wave velocity and ϕ the disturbance stream function.

The present work is very closely related to the works of Rayleigh (1880) [17], Fjørtoft (1935) [10], Tollmein (1935) [18], Fredrichs (1942) [11], Howard (1964) [13] and Banerjee et al. (1995, 2000) [5-7]. We present these in the form of the following mathematical theorems:

Rayleigh's Theorem [17] on Point of Inflexion

If $\left(\phi,c,\alpha^2\right)$ *with* $c_i>0$ *is a solution of Rayleigh equation (1.1) and the boundary conditions (1.2) then*

$$U''=0$$

for some $z=z_s$ *such that* $z_1<z_s<z_2$.

Fjørtoft's stronger version [10] of the theorem on Point of Inflexion

If $\left(\phi,c,\alpha^2\right)$ *with* $c_i>0$ *is a solution of Rayleigh equation (1.1) and the boundary conditions (1.2) then*

$$U''(U-U_s)<0$$

for some $z=z_p$ *such that* $z_1<z_p<z_2$*, where* $U_s=U(z_s)$ *and* $U''=0$ *at* $z=z_s$.

Banerjee and Shandil's more stronger version [6] of the theorem on Point of Inflexion

If (ϕ,c) is a nontrivial non singular solution of the eigenvalue problem (for c) Rayleigh equation (*1.1*) and the boundary conditions (*1.2*) with $D^2U=0$ at $z=z_s\in(z_1,z_2)$ and $K(z)=-\frac{U''}{(U-U_s)}$ satisfies condition of integrability and $0\le K(z)\le\frac{\pi^2}{(z_2-z_1)^2}$ over $[z_1,z_2]$ then c_i must be zero.

Tollmien-Friedrichs' Theorem [18, 11] on Existence of Neutrally Stable Wave

If $K(z) = -\dfrac{U''}{(U-U_s)}$ *is integrable and satisfies* $K(z) > \dfrac{\pi^2}{(z_2 - z_1)^2}$ *in* $z_1 \le z \le z_2$ *, then there exist a solution* (ϕ, c, α^2) *of equation (1.1) and (1.2) and is given by*

$$\phi = \phi_s,$$
$$c = U(z_s) = U_s = c_s$$

and

$$\alpha^2 = \alpha_s^2,\ \alpha_s > 0,$$

where $z_1 < z_s < z_2$ *such that* $U'' = 0$ *at* $z = z_s$ *and* $-\alpha_s^2$ *is the least eigenvalue and* ϕ_s *is the corresponding eigensolution of the real non-singular Sturm-Liouville problem (the Rayleigh equation in the context)*

$$D^2 f + K(z) f - \alpha^2 f = 0$$

together with boundary condition (12), and is given by the variational principle

$$-\alpha_s^2 = \left[\frac{\int_{z_1}^{z_2} \left\{(Df)^2 - K(z) f^2\right\} dz}{\int_{z_1}^{z_2} f^2 dz}\right]_{\min}.$$

Tollmien-Howard Theorem [18, 13] on Existence of ☺¤☺$_s$ -Neutral Stability

If (ϕ, c, α^2) *with* $c_i > 0$ *is a solution of equation (1.1) and (1.2) and* $K(z)$ is integrable and satisfies $K(z) > \dfrac{\pi^2}{(z_2 - z_1)^2}$ in $z_1 \le z \le z_2$ *, then* $0 < \alpha < \alpha_s$ *.*

Banerjee-Shandil and Kanwar's Theorem [5] on Existence of $\alpha^2 \ge \max_{z_1 \le z \le z_2} K(z)$ -Neutral Stability

If (ϕ, c, α^2) *with* $c_i > 0$ *is a solution of equation (11) and (12) and* $K(z)$ is integrable in $z_1 \le z \le z_2$ *, then*

$$\alpha_s^2 \le \max_{z_1 \le z \le z_2} K(z).$$

In the next section, we will first show that the criterion of Banerjee & Shandil [7] can be modified and made more accurate by incorporating in it the wave length effect of the perturbation waves which is missing in the earlier works. Next we will establish a necessary criterion for existence of neutrally stable non singular waves in terms of known parameters i.e. basic velocity distribution U and thickness $|z_2 - z_1|$ of the flow field unifying the Tollmien-Howard [18], [12] Theorem and Banerjee *et al* [5] results.

Mathematical Analysis

Proposition 1: *A necessary condition for the existence of non-trivial, non-singular solution* (ϕ, c, α^2) *of the double eigenvalue problem in* $c = c_r + ic_i$ *with* $c_i > 0$ *and prescribed by the Rayleigh equation (1.1) and associated boundary condition (1.2) is that the following integral relations*

$$\int_{z_1}^{z_2} \left(|D\phi|^2 + \alpha^2 |\phi|^2\right) + \int_{z_1}^{z_2} \frac{U''(U - c_r)}{|U - c|^2} |\phi|^2 = 0 \tag{2.1}$$

and

$$\int_{z_1}^{z_2} \frac{U''}{|U - c|^2} |\phi|^2 = 0 \tag{2.2}$$

must hold.

Proof: Multiplying equation (*1.1*) by ϕ^* (the complex conjugate of ϕ) throughout and integrating the resultant equation over the domain of z with the help of boundary conditions (*12*), we get

$$\int_{z_1}^{z_2} \left(|D\phi|^2 + \alpha^2 |\phi|^2\right) + \int_{z_1}^{z_2} \frac{U''}{U - c} |\phi|^2 = 0. \tag{2.3}$$

Equating the real and imaginary parts of above equation (23) and using the fact that $c_i > 0$, we obtain the required results-

$$\int_{z_1}^{z_2}\left(|D\phi|^2+\alpha^2|\phi|^2\right)+\int_{z_1}^{z_2}\frac{U''(U-c_r)}{|U-c|^2}|\phi|^2=0$$

and

$$\int_{z_1}^{z_2}\frac{U''}{|U-c|^2}|\phi|^2=0.$$

Proposition 2: *A necessary condition for the existence of non-trivial, non-singular solution (ϕ, c, α) of the double eigenvalue problem in $c = c_r + ic_i$ with $c_i > 0$ and prescribed by the Rayleigh equation (11) and associated boundary condition (12) is that the following integral relations*

$$\int_{z_1}^{z_2}\left(|D\phi|^2+\alpha^2|\phi|^2\right)+\int_{z_1}^{z_2}\frac{U''(U-U_s)}{|U-c|^2}|\phi|^2=0 \tag{2.4}$$

and

$$\int_{z_1}^{z_2}\left(\left|D^2\phi\right|^2+2\alpha^2|D\phi|^2+\alpha^4|\phi|^2\right)-\int_{z_1}^{z_2}\frac{U''^2}{|U-c|^2}|\phi|^2=0 \tag{2.5}$$

must hold.

Proof Now multiplying equation (2.2) with the constant term (c_r-U_s) and adding it into equation (2.1), we derive

$$\int_{z_1}^{z_2}\left(|D\phi|^2+\alpha^2|\phi|^2\right)+\int_{z_1}^{z_2}\frac{U''(U-U_s)}{|U-c|^2}|\phi|^2=0,$$

which is the integral relation (2.4).
Now, equation (1.1) can be rewritten as

$$D^2\phi-\alpha^2\phi=\frac{U''}{U-c}\phi,$$

which gives out

$$\left(D^2\phi-\alpha^2\phi\right)\left(D^2\phi^*-\alpha^2\phi^*\right)=\left|\frac{U''}{U-c}\phi\right|^2$$

$$\left|D^2\phi\right|^2-\alpha^2\left(\phi D^2\phi^*+\phi^*D^2\phi\right)+\alpha^4|\phi|^2=\frac{U''^2}{|U-c|^2}|\phi|^2.$$

Integrating above equation over the domain of $z\in[z_1,z_2]$, we have

$$\int_{z_1}^{z_2}\left|D^2\phi\right|^2dz-2\alpha^2\int_{z_1}^{z_2}\left(\phi D^2\phi^*+\phi^*D^2\phi\right)dz+\alpha^4\int_{z_1}^{z_2}|\phi|^2dz=\int_{z_1}^{z_2}\frac{U''^2}{|U-c|^2}|\phi|^2dz \tag{2.6}$$

Integrating second integral by parts and using boundary condition (*1.2*), we obtain the integral relation (2.5).

$$\int_{z_1}^{z_2}\left(\left|D^2\phi\right|^2+2\alpha^2|D\phi|^2+\alpha^4|\phi|^2\right)-\int_{z_1}^{z_2}\frac{U''^2}{|U-c|^2}|\phi|^2=0.$$

Lemma 1: $\int_{z_1}^{z_2}\left|D^2\phi\right|^2dz\ge\frac{\pi^2}{(z_2-z_1)^2}\int_{z_1}^{z_2}|D\phi|^2dz.$

Proof: We have

$$\int_{z_1}^{z_2} |D\phi|^2 dz = -\int_{z_1}^{z_2} \phi^* D^2 \phi dz, \qquad \text{since } \phi(z_1) = 0 = \phi(z_2)$$

$$\leq \left| \int_{z_1}^{z_2} \phi^* D^2 \phi dz \right|$$

$$\leq \int_{z_1}^{z_2} \left| \phi^* D^2 \phi \right| dz$$

$$\leq \int_{z_1}^{z_2} \left| \phi^* \right| \left| D^2 \phi \right| dz$$

$$\leq \int_{z_1}^{z_2} |\phi| \left| D^2 \phi \right| dz.$$

Using Schwartz inequality, we get

$$\int_{z_1}^{z_2} |D\phi|^2 dz \leq \sqrt{\int_{z_1}^{z_2} |\phi|^2 dz} \sqrt{\int_{z_1}^{z_2} \left| D^2 \phi \right|^2 dz}.$$

Using Rayleigh-Ritz inequality

$$\int_{z_1}^{z_2} |D\phi|^2 dz \geq \frac{\pi^2}{(z_2 - z_1)^2} \int_{z_1}^{z_2} |\phi|^2 dz \tag{2.7}$$

(since $\phi(z_1) = 0 = \phi(z_2)$), we get

$$\int_{z_1}^{z_2} |D\phi|^2 dz \leq \frac{(z_2 - z_1)}{\pi} \sqrt{\int_{z_1}^{z_2} |D\phi|^2 dz} \sqrt{\int_{z_1}^{z_2} \left| D^2 \phi \right|^2 dz}.$$

That is

$$\int_{z_1}^{z_2} \left| D^2 \phi \right|^2 dz \geq \frac{\pi^2}{(z_2 - z_1)^2} \int_{z_1}^{z_2} |D\phi|^2 dz. \tag{2.8}$$

Proposition 3: *A necessary condition for the existence of non trivial, non singular solution* (ϕ, c, α^2) *of the double eigen value problem in* $c = c_r + ic_i$ *with* $c_i > 0$ *and prescribed by the Rayleigh equation (1.1) and associated boundary condition (1.2) is*

$$U''^2 + \left\{ \frac{\pi^2}{(z_1 - z_2)^2} + \alpha^2 \right\} U''(U - U_s) \geq 0 \tag{2.9}$$

in some non-zero measurable subset of $[z_1, z_2]$.

Proof: Multiplying (2.4) with $\frac{\pi^2}{(z_1 - z_2)^2} + \alpha^2$ and subtracting from equation (2.5), we get

$$\left[\int_{z_1}^{z_2} \left| D^2 \phi \right|^2 dz - \frac{\pi^2}{(z_1 - z_2)^2} \int_{z_1}^{z_2} |D\phi|^2 dz \right] + \alpha^2 \left[\int_{z_1}^{z_2} |D\phi|^2 dz - \frac{\pi^2}{(z_1 - z_2)^2} \int_{z_1}^{z_2} |\phi|^2 dz \right]$$

$$- \int_{z_1}^{z_2} \frac{U''^2 + \left\{ \frac{\pi^2}{(z_1 - z_2)^2} + \alpha^2 \right\} U''(U - c_r)}{|U - c|^2} |\phi|^2 dz = 0$$

Now by Rayleigh-Ritz inequality (2.8) and inequality (2.9) the first two brackets must have non-negative value, thus

$$\int_{z_1}^{z_2} \frac{U''^2 + \left\{ \frac{\pi^2}{(z_1 - z_2)^2} + \alpha^2 \right\} U''(U - U_s)}{|U - c|^2} |\phi|^2 dz \geq 0 \tag{2.10}$$

i.e.

$$U''^2+\left\{\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}U''(U-U_s)\geq 0$$

in some non-zero measurable subset of $[z_1, z_2]$.

Theorem 1: *A necessary condition for the existence of non-trivial, non-singular solution* (ϕ, c, α^2) *of the double eigen value problem in* $c=c_r+ic_i$ *and prescribed by the equation (1.1) and (1.2) with* $c_i>0$ *and* $K(z)=-\frac{U''}{(U-U_s)}$ *is positive and integrable in the interval* $z_1\leq z\leq z_2$ *, then*

$$K(z)\geq\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2 \tag{2.11}$$

in some non-zero measurable subset of $[z_1, z_2]$.

Proof: The proof of the theorem follows from the equation (2.10)

$$\frac{U''^2}{K(z)}\left\{K(z)-\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}=U''^2+\left\{\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}U''(U-U_s).$$

Corollary 1: *A necessary condition for the existence of non-trivial, non-singular solution* (ϕ, c, α^2) *of the double eigenvalue problem in* $c=c_r+ic_i$ *and prescribed by the equation (1.1) and (1.2) with* $c_i>0$ *and* $K(z)$ *is positive and integrable in the interval* $z_1\leq z\leq z_2$ *, then*

$$\alpha^2\leq[K(z)]_{\max}-\frac{\pi^2}{(z_1-z_2)^2} \tag{2.12}$$

Proof: The proof directly follows from the equation (2.11).

Theorem 2: *A necessary condition for the existence of non-trivial, non-singular solution* (ϕ, c, α^2) *for the neutrally stable (i.e.* $c_i=0$ *) wave and prescribed by the equation (1.1) and (1.2) with* $K(z)$ *is integrable in the interval* $z_1\leq z\leq z_2$ *, then*

$$K(z)\geq\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2. \tag{2.13}$$

Proof: For the neutral stable mode ($c_i=0$) $U''=0$ where $U=c=c_r=U_s$. thus the Rayleigh equation (*1.1*) becomes

$$(D^2-\alpha^2)\phi+K(z)\phi=0 \tag{2.14}$$

Multiplying equation (2.14) with ϕ and integrating it over the domain of z using boundary condition (1.2), we have

$$\int_{z_1}^{z_2}\left[(D\phi)^2+\alpha^2\phi^2-K(z)\phi^2\right]dz=0 \tag{2.15}$$

now using Rayleigh-Ritz inequality as ϕ vanishes on boundary, we have the required result.

Corollary 2: *A necessary condition for the existence of non-trivial, non-singular solution* (ϕ, c, α^2) *of the neutral stable modes and prescribed by the equation (2.14) and (1.2) is*

$$\alpha^2\leq[K(z)]_{\max}-\frac{\pi^2}{(z_1-z_2)^2}. \tag{2.16}$$

Proof: The proof directly follows from the equation (2.15).

Theorem 3: *A necessary condition for the existence of non trivial, non singular solution* (ϕ, c, α^2) *of the double eigen value problem in* $c=c_r+ic_i$ *with* $c_i>0$ *and prescribed by the Rayleigh equation (1.1) and associated boundary condition (1.2) is that the following inequalities*

$$\left\{\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}c_i \le |U''|_{\max},$$

$$\alpha c_i \le \frac{|z_1-z_2|}{\sqrt{2}\pi}|U''|_{\max},$$

$$\alpha c_i \le \frac{|U''|_{\max}}{\alpha},$$

and

$$\left\{\frac{2\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}c_i^2 \le \frac{|U''|_{\max}^2}{\alpha^2}$$

must hold.

Proof: Making use of equation (2.5), Rayliegh-Ritz inequality (2.8) and inequality (2.9); we get

$$\int_{z_1}^{z_2}\left[\left\{\frac{\pi^2}{(z_1-z_2)^2}\right\}^2|\phi|^2+2\alpha^2\left\{\frac{\pi^2}{(z_1-z_2)^2}\right\}|\phi|^2+\alpha^4|\phi|^2\right]-\int_{z_1}^{z_2}\frac{U''^2}{|U-c|^4}|\phi|^2\le 0$$

i.e.

$$\int_{z_1}^{z_2}\left[\left\{\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}^2-\frac{U''^2}{|U-c|^2}\right]|\phi|^2\le 0.$$

Therefore if above integral relation is satisfied then we must have

$$\left\{\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}^2 \le \left[\frac{U''^2}{|U-c|^2}\right]_{\max}.$$

Now using the fact $c_i^2 \le |U-c|^2$, we get

$$\left\{\frac{\pi^2}{(z_1-z_2)^2}+\alpha^2\right\}^2 c_i^2 \le |U''|_{\max}^2,$$

$$\alpha^2 c_i^2 \le \frac{(z_1-z_2)^2}{2\pi^2}|U''|_{\max}^2,$$

$$\alpha^2 c_i^2 \le \frac{|U''|_{\max}^2}{\alpha^2}$$

and

$$\left(\frac{2\pi^2}{(z_1-z_2)^2}+\alpha^2\right)c_i^2 \le \frac{|U''|_{\max}^2}{\alpha^2}.$$

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