A SIR model with time-dependent contact rate for influenza H1N1

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*Abstract :*The aim of this paper is to develop, analyze and simulate a control strategy through SIR mathematical model for H1N1 with the help of periodic contact rate. Since H1N1 spreads through contact with an infected individual, therefore an effective contact rate has been modelled to minimize the effects of the disease on the society. The effective contact rate, which has been taken as a parameter earlier has been considered as a function of time t for the present study. Numerical simulations of the model have been performed with the help of fourth-order Runge Kutta method to illustrate our results. With the help of simulation, spread and control of disease has been demonstrated for different values of effective contact rate and action time, which have been found playing crucial role in controlling the spread of the disease, and differences have been displayed with the help of the graphs.

***IndexTerms*-action time; Influenza A (H1N1); effective contact rate; SIR model.**

I. INTRODUCTION

There are various types of epidemics such as HIV/AIDS, measles, rubella and the most recent is H1N1. The **Hemagglutinin Type 1 and Neuraminidase Type 1 (H1N1) is a pandemic disease spreading worldwide [15] and** a major public – health problem from 2009 in the world. The first case of H1N1 reported on 9 April 2009 in Mexico, after which the World Health Organization declared it a pandemic in June 2009 [15]. The H1N1 flu virus spreads in the same way as other regular seasonal influenza viruses spread, i.e. from person-to-person. The H1N1 flu is also known as swine flu because the people who suffered from it first time were in direct contact with pigs and also the first detected virus contains a combination of genes from pigs, birds, and humans. The exact incubation period of H1N1 is unknown and could range from 1 to 7 days. Persons with novel H1N1 flu virus infection should be considered potentially contagious for up to 7 days after the onset of illness. In the mathematical biological literature, several mathematical models have been proposed to model the spread of infectious diseases such as SIS [8], SIR ([8], [12], [13], [14]), SEIR [9], SEIRS [17], SVEIR [1] (where S, V, E, I and R denotes the population of susceptible, vaccinated, exposed, infected and recovered individuals respectively). In classical epidemiology, a critical factor is known as 'mass action principle' which states that the course of an epidemic depends on the contact rate between susceptible and infected individuals. Also, the net rate at which infection is acquired is proportional to the numbers of encounters between susceptible and infected individuals [11]. The constant of proportionality, denoted by β has been termed as the transmission coefficient [8]. As it has been reported that, H1N1 flu spreads due to close contact with infected individuals [15], therefore this transmission coefficient can play a vital role in controlling the spread of the disease. Transmission coefficient has been allotted a fixed value for simulation of the model in all the previous models for H1N1 ([6], [9]). Since the transmission coefficient depends on the population size, therefore modeling of effective contact rate may prove helpful for health agencies in controlling the spread of disease. It is well-known fact that many diseases, such as influenza, measles, whooping cough, etc., exhibit seasonal (periodic) fluctuations, ([7], [10], [16]) and hence spread of the disease depends on the contact rate. Contact rate was considered as a parameter before Juhan Zhang et al. [5] have considered time varying periodic effective contact rate for the control of rabies in China. Since contact rate is one of the major factors in the spread of the disease, the present paper aims to develop a SIR model for H1N1 with the induction of periodic effective contact rate function. Also, a new term, "Action time" is also introduced in the study which can be defined as "A time in which we can control the spread of infection from infected to susceptible individuals by various means such as by increasing the immunity of infected and susceptible, vaccination of both susceptible and infected, quarantization of infected, yoga etc."

In the present article, we have devolved the SIR model for H1N1 after including effective contact rate function and performed the simulations based on the data of human H1N1 cases reported by Morocco in 2009 [6].

Further, the section wise detail of the paper is as follows: Section 2 consists of a formulation of SIR model for H1N1. An effective contact rate function is described and modelled in section 3. Basic properties of the model are given in Section 4. Numerical simulations are in section 5. Finally, the conclusion is summarized in section 6.

II A SIR MODEL FOR H1N1

In this section, we have developed a SIR model for the transmission of H1N1. For this, the total population is divided into three compartments: susceptible $S(t)$, infective I(t) and recovered$R(t)$. The motive is to control the spread of H1N1 using mathematical model, for which we have assumed that susceptible individuals can be infected only through contact with infectious individuals. Therefore, instead of considering β as a parameter, we have formulated an effective contact rate which is a function of time t. The total population size is $N(t) = S(t) + I(t) + R(t)$. The progression of population among three different compartments S, I and R can be understood from the figure given below:
After incorporating these changes in [6], the rate of change of the population in each compartment is given by the following nonlinear system of differential equations:

$$\frac{dS}{dt} = \Delta - \mu S - \beta(t) S \frac{I}{N} \quad (2.1)$$

$$\frac{dI}{dt} = \beta(t) S \frac{I}{N} - (\mu + d + \delta) I, \quad (2.2)$$

$$\frac{dR}{dt} = \delta I - \mu R \quad (2.3)$$

where $S(0) = S_0, I(0) = I_0, R(0) = R_0, t \geq 0$ and $\beta(t) > 0$ is an effective contact rate function. The effective contact rate function $\beta(t)$ is described in next section.
The descriptions of above model parameters are listed in Table 1 [5].

III DESCRIPTION AND MODELLING OF EFFECTIVE CONTACT RATE FUNCTION$\boldsymbol{\beta(t)}$

In this section, we have formulated effective contact rate function.It is well-known fact that the probability of getting a disease is not constant at any point of time. Some diseases are seasonal, such as the H1N1, which is more prevalent during winter. Also, thetransmission rate of H1N1 is very fast and also in real world phenomena, the effective contact between individuals is not a just matter of perception. For example, in the classroom, office etc., we use contact rate as a periodic function of time t. To formulate effective contact rate function, following assumptions have been made:

1. It has been observed from the past that spread of H1N1 increases with time up tocertain period of time.
2. Contact between infected and susceptible individuals is held responsible for the spread of disease. Also, contact between infected and susceptible individuals is in a periodic manner, for example, in the classroom, office etc. and therefore, effective contact rate should increase with time in a periodic manner.
3. Also, it is considered that contact rate between susceptible and infected individuals cannot be completely zero at any time t.
4. A force of infection(F) with periodically ('seasonal') varying contact rate has been considered i.e.
   $F = \frac{\beta(t) I}{N}, \beta(t+T) = \beta(t)$ (3.1)
   with period T equal to one year.

Hence effective contact rate function $\beta(t)$ has been modeled as follows

$$\beta(t) = \frac{G(t)+c}{b}, 2 \leq c \leq 7, b = c + 2 \; and \; t \geq 0 \quad (3.2)$$

where $G(t)$ is also a Periodic function of time with period$T$.
For the purpose of simulations, effective contact rate function $\beta(t)$ has been formulated as follows

$$\beta(t) = \frac{Sin^2 t + c}{b}, 2 \leq c \leq 7, b = c + 2 \; and \; t \geq 0 \quad (3.3)$$

where b is the spread controlling parameter to minimize the infection and c is an action time, "A time in which we can control the spread of infection from infected to susceptible individuals by various means such as by increasing the immunity of infected and susceptible, vaccination of both susceptible and infected, quarantization of infected, yoga etc.". The value of c has been assumed greater and equal to two days because we have assumed that it will take at least two days to control the spread of disease. We have also considered that ideally the authority should take maximum 7 days which is also the incubation period of H1N1, to initiate the preventive measures and awareness about the disease. However, the value of c may be more than 7 days.

IV BASIC PROPERTIES OF THE MODEL

The model system (Equations$(2.1) - (2.3)$) monitors populations, it is assumed that all state variables and parameters of the model are nonnegative i.e. $(S, I, R) \in \mathbb{R}^3_+$ and $\Delta, \mu, d, \delta \geq 0$.

Theorem 1: The variables of the model (Equations$(2.1) - (2.3)$) are non-negative at all time.
Proof: Let$t_* = \sup\{t > 0 : S > 0, I > 0, R > 0 \in (0, t)\}$. Thus, $t_* > 0$. It follows the equation (2.1) that

$$\frac{dS}{dt} = \Delta - \mu S - \beta(t) S \frac{I}{N}$$

which can be re-written as,

$$\frac{d}{dt}\left[S(t)e^{\left\{\mu t+\frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}}\right] = \Delta\, e^{\left\{\mu t+\frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}} \quad (4.1)$$

Hence,

$$S(t_*)e^{\left\{\mu t_*+\frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}} - S(0) = \int_0^{t_*} \Delta\, e^{\left\{\mu \tau+\frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}}d\tau,$$

So that

$$S(t) = S(0)e^{-\left\{\mu t_*+\frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}} + \left[e^{-\left\{\mu t_*+\frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}}\right]\int_0^{t_*} \Delta\, e^{\left\{\mu \tau+\frac{1}{N}\int_0^t \beta(y)I(y)dy\right\}}d\tau > 0.$$

Similarly $I > 0$ and $R > 0$.

Lemma 1: The closed set $D = \{(S, I, R) \in \mathbb{R}^3_+ : N \leq \frac{\Delta}{\mu}\}$ is positive- invariant.

Proof:

The rate of change of the total population, obtained by adding Equations $(2.1) - (2.3)$ is given by

$$\frac{dN}{dt} = \Delta - \mu N - dI. \quad (4.2)$$

Since $\frac{dN}{dt} \leq \Delta - \mu N$, it follows that $\frac{dN}{dt} \leq 0$ if $N \geq \frac{\Delta}{\mu}$. Thus, a standard comparison theorem for ODE can be used to show that $N(t) \leq \frac{\Delta}{\mu} + (N(0) - \frac{\Delta}{\mu})e^{-\mu t}$. In particular, $N(t) \leq \frac{\Delta}{\mu}$ if $N(0) \leq \frac{\Delta}{\mu}$. Thus, the region D is positively- invariant. Further, if $N(0) > \frac{\Delta}{\mu}$, then either the solution enters in D finite time, or $N(t)$ approaches $\frac{\Delta}{\mu}$ asymptotically. Hence region D attracts the all solutions in $\mathbb{R}^3_+$.

The system (Equations $(2.1) - (2.3)$) is continuous and its derivative implies that solutions exist and is unique. Since solutions approach lies in D they are bounded and hence exist for $t \geq 0$. Therefore, the model is epidemiologically and mathematically well posed.

V NUMERICAL SIMULATION AND DISCUSSION

Numerical simulation of the model equations (2.1 - 2.3) has been performed using Fourth order Runge- Kutta method ([2], [3], and [4]) in Matlab 2012b. The values of the parameters have been taken from table 1 and the values of S_0, I_0 and R_0 have been taken from [6]. Graphs have been drawn after simulation of the model for which explanations are given subsequently below.

Figure 2, Figure 3 and Figure 4 shows the population of susceptible, infected and recovered individuals for c $= 3, 5$ and 7 respectively with respect to time:

1. Figure 2(a), 3(a) and 4(a) show that the population of susceptible individuals decreases as time increases and after approximately 50, 45 and 40 days respectively the population of susceptible individuals becomes constant.
2. Figure 2(b), 3(b) and 4(b) show the population of infected individuals with respect to time for different values of c as stated above.
3. Figure 2(c), 3(c) and 4(c) show that the population of recovered individuals increases with respect to time and after approximately 50, 45 and 40 days respectively, the population of recovered individuals becomes constant.
4. And Figure 2(d), 3(d) and 4(d) show the behaviour of susceptible, infected and recovered individuals as time increases.

And also Figure 2, Figure 3 and Figure 4 show that the populations of susceptible infected and removed individuals converge asymptotically to an endemic equilibrium state with respect to time.

Action time plays an important role in controlling the infection of disease. The population of infected individuals at various values of action time $c = 3, 5, 7$ is shown in figure 5. Clearly we can observe from figure 5 that population of infected individuals increases as the value of action time increases i.e. more number of days will be taken to initiate corrective measures which implies that more number of people will be infected. It can be understood that if time take for preventive measures, i.e. action time, is increased then infected population will increase. Also it can be observed that less number of days for action time implies less infected population. Preventive measures may include various methods such as the decreased contact rate and quarantinization of the infected individuals and many more.

Figure 6 and 7 show the population of infected individuals at different values of I_0 (initially infected individuals) for c $= 5$ and 3. It can be observed from figure 6 and 7 that the population of infected individuals at different values of I_0 is approximately same for both the values of c.

It is also observed from figure 6 that for more number of initially infected individuals (i.e. $I_0 = 3000$), the disease will take less time in reaching to its peak while as the number of initially infected individuals decreases ($I_0 = 30,\ I_0 = 300$), the disease will acquire its peak after more number of days.

From figure 7 it can be observed that for c $= 3$ i.e. for decreased action time, the number of infected individuals is less for different values of I_0 in comparison of infected individuals for c $= 5$. Also, it can be seen from figure 7 that for decreased action time, the infection will stay for less number of days, which indicates that severity of disease may be controlled as action time decreases.

Figure 8 shows the transmission rate of H1N1 with time at different values of action time. Cleary, it can be seen that as time increases, the area under the curve for transmission rate with small action time is low in comparison of greater values of action

time. Action time may be utilized to take preventive method to control the disease and also increasing the immunity of the susceptible to avoid them to become infected and of the infected to make a fast recovery.
Figure 9 shows the population of susceptible, infected and recovered individuals respectively when effective contact rate is a parameter. For simulation, we take parameters value from table 1. Clearly, Figure 2 and 9 shows the difference in the population of susceptible, infected and recovered individuals when effective contact rate has been taken as a function and parameter respectively.

Figure 10 shows the difference of infected population between effective contact rate as a parameter and a periodic function. From the figures, it is depicted that infected population is less when effective contact rate is considered as a periodic function of time in place of a parameter. From this, it can be concluded that spread of the disease may be controlled by controlling the effective contact rate which is responsible for the spread of disease. Hence, modelling of effective contact rate along with action time as a function can be proved very helpful in throwing some light on the control of the spread of the disease which in result may save many lives.

Hence, thelength of action time may play a crucial role in controlling the spread of disease like H1N1 and may be helpful in preventing the susceptible from becoming newly infected individuals. Preventive measures should be taken within action time to control the endemic behaviour of the disease.

V CONCLUSION

As we know that for understanding the dynamics of an epidemic and also for planning and evaluating interventions, mathematical modelling is a valuable tool. In this paper, we have presented a SIR model by considering effective contact rate as a periodic function of time to control the spread of the H1N1 infection with the inclusion of new term, action time. From the simulation of the model, it has been demonstrated that action time helps to control the spread of H1N1 infection. It can be concluded that more the value of action time, more individuals will become infected since action time is the duration of time taken to initiate preventive measures to control the spread of the disease. Therefore, action time may further be proved extremely helpful in saving money spent on medicinal treatment and lives of individuals if a critical value may be found out to take appropriate preventive actions for a particular disease.

FiguresandTables

Table 1: Parameters description and values used in simulations

Parameter	Description	Value	Source
Δ	Recruitment rate	1175 per day	Hattaf et al. (2009)
μ	Natural mortality rate	3.9139×10^{-5} per day	Hattaf et al. (2009)
d	Disease-induced mortality rate	0.067 per day	Hattaf et al. (2009
δ	Recovery rate	0.2 per day	Hattaf et al. (2009)
c	Action time	Incubation Period of disease 2 to 7 days	Assumption [15]
b	Spread controlling Parameter	$(c+2)$ days	Estimation
β	Effective contact rate	0.80127417 per day	Hattaf et al. (2009)

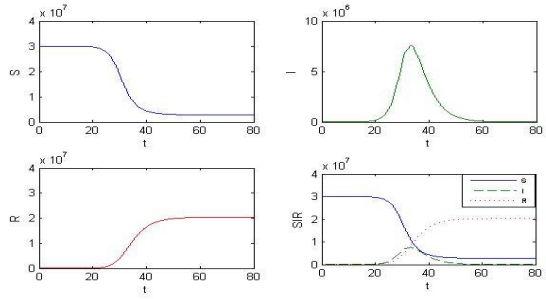

Figure 2: SIR graph at $c = 3$

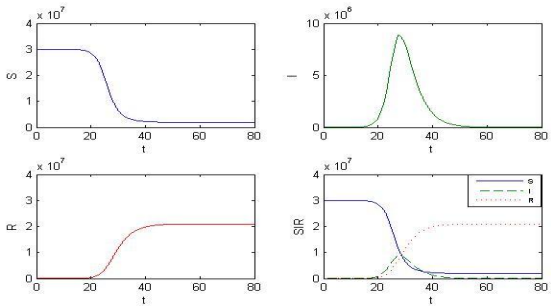

Figure 3: SIR graph at $c = 5$

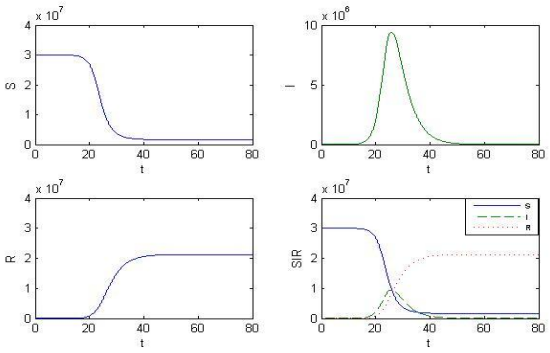

Figure 4: SIR graph at $c = 7$

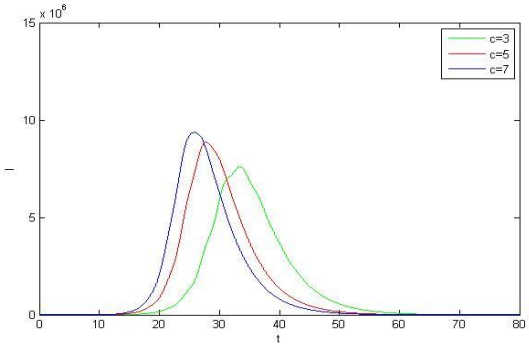

Figure 5: Number of infected individuals at various values of c.

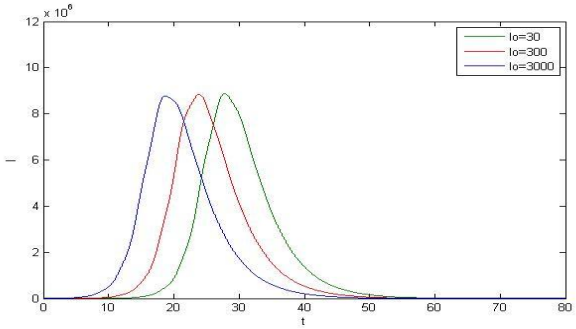

Figure 6: Number of infected individuals for different values of I_0 at $c = 5$.

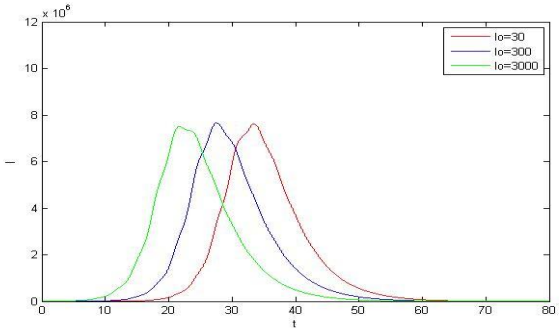

Figure 7: Number of infected individuals for different values of I_0 at $c = 3$.

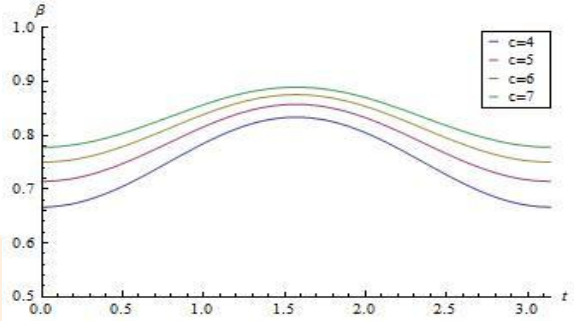

Figure 8: Effective contact rate $\beta(t)$ at different values of c in a year.

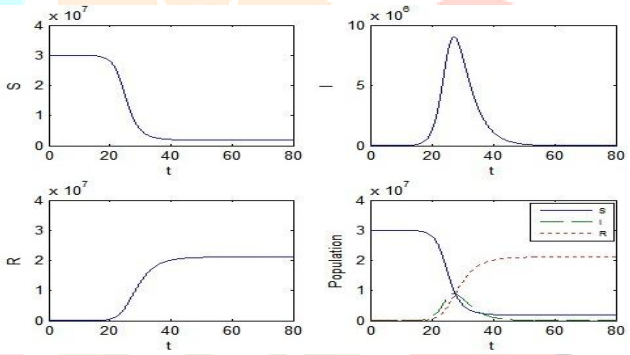

Figure 9: SIR graphs at effective contact rate as a parameter

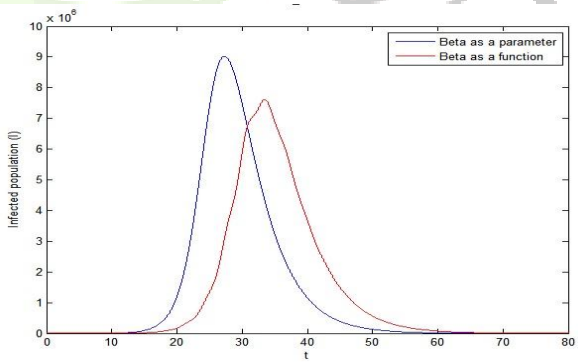

Figure 10: Infected population at effective contact rate as a function and parameter.

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